Cambridge Assessment
Admissions Testing

STEP Mark Schemes 2017

Mathematics

STEP 9465/9470/9475

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STEP Mathematics (9465, 9470, 9475)

Introduction

These mark schemes are published as an aid for teachers and students, and indicate the requirements of the examination. It shows the basis on which marks were awarded by the Examiners and shows the main valid approaches to each question. It is recognised that there may be other approaches; if a different approach was taken by a candidate, their solution was marked accordingly after discussion by the marking team. These adaptations are not recorded here.

All Examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the Report on the Examination.

Admissions Testing will not enter into any discussion or correspondence in connection with this mark scheme.

Marking notation

NOTATION	MEANING	NOTES
M	Method mark	For correct application of a **M**ethod.
dM or m	Dependent method mark	This cannot be earned unless the preceding **M** mark has been earned.
A	Answer mark	**M0** ⇒ **A0**
B	Independently earned mark	Stand alone for "right or wrong".
E	B mark for an explanation	
G	B mark for a graph	
ft	Follow through	To highlight where incorrect answers should be marked as if they were correct.
CAO or **CSO** **Sometimes written as A***	Correct Answer/Solution Only	To emphasise that **ft** does not apply.
AG	Answer Given	Indicates answer is given in question.

Question 1

(i) $u = x\sin x + \cos x$

$\frac{du}{dx} = \sin x + x\cos x - \sin x$ **M1 A1**

$= x\cos x$

$\int \frac{x}{x\tan x+1}dx = \int \frac{x\cos x}{x\sin x+\cos x}dx$ **M1 A1**

$= \int \frac{1}{u}du$

$= \ln|u| + c$ **A1**

$\therefore \int \frac{x}{x\tan x+1}dx = \ln|x\sin x + \cos x| + c$ **M1**

$\int \frac{x}{x\cot x-1}dx = \int \frac{x\sin x}{x\cos x-\sin x}dx$ **M1**

Let $u = x\cos x - \sin x$

$\frac{du}{dx} = \cos x - x\sin x - \cos x$

$= -x\sin x$ **M1 A1**

$\int \frac{x}{x\cot x-1}dx = \int \frac{-1}{u}dx = -\ln|u| + c$

$\therefore \int \frac{x}{x\cot x-1}dx = -\ln|x\cos x - \sin x|$ **A1**

(ii) Let $u = x\sec^2 x - \tan x$ **M1 A1**

$\frac{d}{dx}(\sec^2 x) = 2\sec x\,(\sec x\tan x) = 2\sec^2 x\tan x$ **A1**

$\frac{du}{dx} = \sec^2 x + 2x\sec^2 x\tan x - \sec^2 x = 2x\sec^2 x\tan x$ **A1**

So $\int \frac{x\sec^2 x\tan x}{x\sec^2 x-\tan x}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + c$

$\int \frac{x\sec^2 x\tan x}{x\sec^2 x-\tan x}dx = \frac{1}{2}\ln|x\sec^2 x - \tan x| + c$ **A1**

$\int \frac{x\sin x\cos x}{(x-\sin x\cos x)^2}dx = \int \frac{x\sec^2 x\tan x}{(x\sec^2 x-\tan x)^2}dx$ **M2 A1**

Using same substitution as previous integral:

$= \frac{1}{2}\int \frac{1}{u^2}du = -\frac{1}{2u} + c$ **A1**

$\int \frac{x\sin x\cos x}{(x-\sin x\cos x)^2}dx = -\frac{1}{2(x\sec^2 x-\tan x)} + c$ **A1**

Question 1

An **A1** should be lost if modulus signs or $+c$ are omitted in the final answer for any section, but only on the first occasion.

M1	Calculation of $\frac{du}{dx}$
A1	Correct expression
M1	Use of $\tan x = \frac{\sin x}{\cos x}$
A1	Integral simplified and in terms of u
A1	Integration completed correctly in terms of u
M1	Integral rewritten in terms of x
	Subtotal: 6
M1	Rewriting integral in a form ready for substitution
M1	Correct choice of substitution
A1	Correctly differentiated
A1	Correct final answer.
	Subtotal: 4
M1	Choice of a sensible substitution, based on the denominator.
A1	Correct choice
A1	Differentiation of $\sec^2 x$
A1	Correct $\frac{du}{dx}$
A1	Correct final answer.
	Subtotal: 5
M2	Transformation of the integral so that the denominator is similar to the first part.
A1	Correctly transformed.
A1	Correct integral in terms of u
A1	Correct final answer.
	Subtotal: 5

Question 2

(i)	$\int_1^x \frac{1}{t}\,dt = [\ln\lvert t\rvert]_1^x = \ln x$	**M1**
	$\int_1^x 1\,dt = [t]_1^x = x - 1$	
	Therefore, $\ln x \le x - 1$	**A1 AG**
	Over the interval $x \le t \le 1$, $\frac{1}{t} \ge 1$	**M1**
	Taking the integral over the range $x \le t \le 1$ gives the inequality $$-\ln x \ge 1 - x$$	
	Therefore $\ln x \le x - 1$	**A1 AG**
(ii)	$\int_1^x \frac{1}{t^2}\,dt = \left[-\frac{1}{t}\right]_1^x = 1 - \frac{1}{x}$	**M1 A1**
	Therefore, integrating both sides gives	
	$1 - \frac{1}{x} \le \ln x$	**M1 A1**
	and so $\ln x \ge 1 - \frac{1}{x}$ for $x \ge 1$	
	For $0 < x < 1$, integrating $\frac{1}{t^2} \ge \frac{1}{t}$ over the interval $x \le t \le 1$ gives:	
	$\frac{1}{x} - 1 \ge -\ln x$	**M1**
	So $\ln x \ge 1 - \frac{1}{x}$ for $0 < x < 1$ as well and so (**) is true for $x > 0$.	**A1**
	ALTERNATIVE	
	$\frac{d}{dx}\left(1 - \frac{1}{x}\right) = \frac{1}{x^2}$ and $\frac{d}{dx}(\ln x) = \frac{1}{x}$	**M1 A1**
	When $x = 1$, $1 - \frac{1}{x} = 0 = \ln x$	**dM1**
	$\frac{1}{x^2} \ge \frac{1}{x}$ for $x \le 1$	**B1**
	Therefore, since the two sides of the inequality are equal when $x = 1$, the LHS grows more rapidly for $x < 1$ and the RHS grows more rapidly for $x > 1$, the inequality is true.	**G1 E1**
(iii)	$\int_1^y \ln x\,dx = \int_1^y 1 \times \ln x\,dx$	
	$u = \ln x \quad \frac{dv}{dx} = 1$	**M1**
	$\frac{du}{dx} = \frac{1}{x} \quad v = x$	
	$\int_1^y \ln x\,dx = [x \ln x]_1^y - \int_1^y 1\,dx = y\ln y - y + 1$	**A1**
	Integrating (*):	
	For $y > 1$:	**M1**
	$y\ln y - y + 1 \le \frac{y^2}{2} - y + \frac{1}{2}$	
	Therefore $2y\ln y \le (y^2 - 1)$ and so $\frac{\ln y}{y-1} \le \frac{y+1}{2y}$ (since $2(y-1) > 0$)	**A1**
	For $0 < y < 1$:	**M1**
	$y - y\ln y - 1 \le -\frac{y^2}{2} + y - \frac{1}{2}$	
	Therefore $2y\ln y \ge \frac{y^2}{2} - y + \frac{1}{2}$ and so $\frac{\ln y}{y-1} \le \frac{y+1}{2y}$ (since $2(y-1) < 0$)	**A1**
	Integrating (**):	
	For $y > 1$:	**M1**
	$y\ln y - y + 1 \ge y - \ln y - 1$	
	Therefore $(y+1)\ln y \ge 2(y-1)$ and so $\frac{2}{y+1} \ge \frac{\ln y}{y-1}$ (since $(y-1)(y+1) > 0$)	**A1**
	For $0 < y < 1$:	**M1**
	$y - y\ln y - 1 \ge \ln y + 1 - y$	
	Therefore $(y+1)\ln y \le 2(y-1)$ and so $\frac{2}{y+1} \ge \frac{\ln y}{y-1}$ (since $(y-1)(y+1) < 0$)	**A1**

Question 2

M1	Integration of one of the sides of the inequality (indefinite integration OK)
A1	Integration of both sides of the inequality and conclusion reached. (In the case of the RHS an alternative would be a clear explanation in terms of area of rectangle)
M1	Statement of the inequality for this range of values for t.
A1	Correctly drawn conclusion.
	Subtotal: 4
M1	Integration of LHS of inequality. (indefinite integration OK)
A1	Integration completed correctly.
M1	Inequality formed by integrating both sides of inequality
A1	Correct deduction of (**) for $x \geq 1$
Marks up to this point can be awarded if there is no consideration of which values of x (**) is shown for.	
M1	Integration of correct inequality for $0 < x < 1$
A1	Conclusion of (**) including clear explanation of how it is shown for whole range.
Note that substituting $\frac{1}{x}$ for x in (*) gives $\ln\frac{1}{x} \leq \frac{1}{x} - 1$, which leads to $-\ln x \leq \frac{1}{x} - 1$ and then to (**) directly, but no marks for this as question requires starting from a different inequality.	
ALTERNATIVE	
M1	Two differentiations.
A1	Correctly completed.
dM1	Consideration of $x = 1$
B1	Correct inequality.
G1	Graph to illustrate that the inequality holds.
E1	Explanation (award the G1 also for a good explanation without the graph sketched.
	Subtotal: 6
M1	Use of integration by parts to integrate $\ln x$ (indefinite integration OK)
A1	Correct integral
M1	Integration of both sides of inequality in case $y > 1$.
A1	Inequality deduced for case $y > 1$.
M1	Integration of both sides of inequality in case $0 < y < 1$.
A1	Inequality deduced for case $0 < y < 1$.
M1	Integration of both sides of inequality in case $y > 1$.
A1	Inequality deduced for case $y > 1$.
M1	Integration of both sides of inequality in case $0 < y < 1$.
A1	Inequality deduced for case $0 < y < 1$.
	Subtotal: 10

Question 3

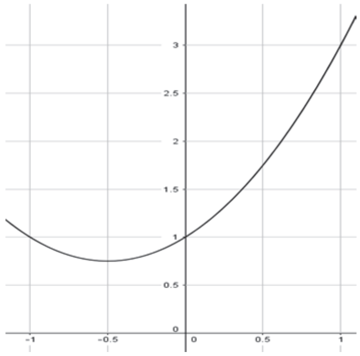
$2y\frac{dy}{dx} = 4a$	**M1**
At P, $\frac{dy}{dx} = \frac{4a}{2(2ap)} = \frac{1}{p}$	**A1**
Therefore the equation of the tangent is: $y - 2ap = \frac{1}{p}(x - ap^2)$	**M1**
$y = \frac{1}{p}x + ap$	**A1**
Similarly, the equation of the tangent at Q is $y = \frac{1}{q}x + aq$	**B1**
Coordinates of R: $\frac{1}{p}x + ap = \frac{1}{q}x + aq$ $qx + ap^2q = px + apq^2$ $(p-q)x = apq(p-q)$ Therefore $x = apq$. [$y = \frac{1}{p}(apq) + ap = a(p+q)$] The coordinates of R are $(apq, a(p+q))$	**M1** **A1 A1**
Other coordinates are: $S(0, ap)$ and $T(0, aq)$	**B1**
Area of RST: Using the edge along the y-axis as the base: Base length $= a(p-q)$ Height $= -apq$ Therefore the area is $\frac{1}{2}a^2pq(q-p)$	**M1 A1** **A1**
Area of OPQ: Trapezium formed by adding horizontals to y-axis from P and Q: Area $= \frac{1}{2}(ap^2 + aq^2)(2ap - 2aq) = a^2(p^2+q^2)(p-q)$	**M2 A1**
Triangles to be removed: (P): Area $= \frac{1}{2}(ap^2)(2ap) = a^2p^3$ (Q): Area $= \frac{1}{2}(aq^2)(-2aq) = -a^2q^3$	**A1 A1**
Therefore area of OPQ= $a^2(p^2+q^2)(p-q) - a^2p^3 + a^2q^3$ Area $= a^2(p^3 + pq^2 - p^2q - q^3 - p^3 + q^3) = a^2pq(q-p)$ Therefore the area of OPQ is twice the area of RST	**M1 A1** **B1**

Question 3

M1	Differentiation.
A1	Gradient correct.
M1	Equation of line through P with this gradient.
A1	Establish given equation.
	Subtotal: 4
B1	Tangent at Q.
	Subtotal: 1
M1	Equating y coordinates.
A1	Solve for x (allow mark if signs are incorrect).
A1	Correct signs.
	Subtotal: 3
B1	Coordinates of other points.
M1	Valid method for area of triangle.
A1	Area correct.
A1	Sign correct.
	Subtotal: 4
M2	Valid method for calculation of the area.
A1	Correct area for one shape that is needed.
A1	Correct area for at least one other shape that is needed.
A1	Correct areas found for all shapes needed.
M1	Combine all areas correctly.
A1	Correct area of the triangle.
B1	All correct and conclusion reached.
	Subtotal: 8

Question 4

(i)



$1 + r + r^2 \equiv \left(r + \frac{1}{2}\right)^2 + \frac{3}{4}$

Therefore minimum point is $\left(-\frac{1}{2}, \frac{3}{4}\right)$

B1
M1
A1
B1
B1

Since $|r| < 1$, upper bound for p is the value when $r = 1$.

Therefore $\frac{3}{4} \leq p < 3$ **B1**

Since $|r| < 1$, there is only one value of r that corresponds to each value of p in the range $1 < p < 3$ (as can be seen on the graph). Therefore values of p in that range determine r (and hence S) uniquely. **B1**

If $\frac{3}{4} < p < 1$ then there are two values of r that satisfy $1 + r + r^2 = p$.

Rearranging $S = \frac{1}{1-r}$ gives $r = 1 - \frac{1}{S}$ **M1**

Substituting:

$1 + \left(1 - \frac{1}{S}\right) + \left(1 - \frac{1}{S}\right)^2 = p$ **M1**

So $3 - \frac{3}{S} + \frac{1}{S^2} = p$ and therefore $(3 - p)S^2 - 3S + 1 = 0$ **A1 AG**

(ii)

$1 + 2r + 3r^2 \equiv 3\left(r + \frac{1}{3}\right)^2 + \frac{2}{3}$ **M1 A1**

At $r = -1$, $1 + 2r + 3r^2 = 2$

At $r = 1$, $1 + 2r + 3r^2 = 6$ **B1**

Therefore values of q in the range $2 \leq q < 6$ determine r, and hence T uniquely. **B1**

If $q = \frac{2}{3}$ then the value of r and hence T is also determined uniquely. **B1**

For $\frac{2}{3} < q < 2$, there are two values of r. **B1**

$(1 - r)^2 = \frac{1}{T}$, so $r = 1 - \frac{1}{\sqrt{T}}$ **M1**

Therefore $q = 1 + 2\left(1 - \frac{1}{\sqrt{T}}\right) + 3\left(1 - \frac{1}{\sqrt{T}}\right)^2$ **A1**

$q = 6 - \frac{8}{\sqrt{T}} + \frac{3}{T}$ and so $(6 - q)T + 3 = 8\sqrt{T}$ **M1**

Squaring:

$(6 - q)^2T^2 + 6(6 - q)T + 9 = 64T$ **A1**

$(6 - q)^2T^2 - (28 + 6q)T + 9 = 0$

Question 4

B1	Correct shape (must include range $\|r\|$<1).
M1	Completing the square (or differentiating) to find minimum.
A1	Correct minimum point.
B1	y-intercept at (0,1).
B1	No intercept with x-axis.
B1	Upper bound for p justified.
B1	Any valid explanation.
M1	Rearrangement of the formula for S.
M1	Substitution into the formula for p.
A1	Correct solution.
	Subtotal: 10
M1	Completing the square (or differentiation to find minimum)
A1	Completion of square done correctly.
B1	Calculation of the value at the two endpoints.
B1	Interval $2 \leq q < 6$ identified.
B1	$q = \frac{2}{3}$ identified.
	Subtotal: 5
B1	Correct interval identified.
M1	Rearrangement of the formula for T.
A1	Substitution into the formula for q.
M1	Simplification into a three term quadratic in $\sqrt{T}$.
A1	Quadratic in T found.
	Subtotal: 5

Question 5

The width of the rectangle is $(s-x)$	
The height of the rectangle is $x\tan\beta$	
Therefore the area is $x(s-x)\tan\beta$	**B1 AG**
The coordinates of R must be $(s, \sqrt{a^2-s^2})$	**B1**
The coordinates of Q must be $(x, x\tan\beta)$	
Therefore $x\tan\beta = \sqrt{a^2-s^2}$ since both must have the same y-coordinate	**M1**
So $s^2 = a^2 - x^2\tan^2\beta$, $s = \sqrt{a^2 - x^2\tan^2\beta}$	**A1**
$2s\frac{ds}{dx} = -2x\tan^2\beta$	**M1**
$\frac{dA}{dx} = (s-x)\tan\beta + x\left(\frac{ds}{dx} - 1\right)\tan\beta$	**M1 A1**
Substituting for $\frac{ds}{dx}$: $\frac{dA}{dx} = (s-x)\tan\beta + x\left(-\frac{x}{s}\tan^2\beta - 1\right)\tan\beta$	**M1**
$\frac{dA}{dx} = (s-2x)\tan\beta - \frac{x^2}{s}\tan^3\beta$	**A1 AG**
ALTERNATIVE (from $s^2 = a^2 - x^2\tan^2\beta$, $s = \sqrt{a^2 - x^2\tan^2\beta}$)	
$A = x\tan\beta\sqrt{a^2 - x^2\tan^2\beta} - x^2\tan\beta$	**dM1**
$\frac{dA}{dx} = \tan\beta\sqrt{a^2 - x^2\tan^2\beta} - 2x\tan\beta - \frac{x\tan\beta(2x\tan^2\beta)}{2\sqrt{a^2-x^2\tan^2\beta}}$	**M1 A1**
Since $\sqrt{a^2 - x^2\tan^2\beta}$, $\frac{dA}{dx} = (s-2x)\tan\beta - \frac{x^2}{s}\tan^3\beta$	**M1 A1**
Stationary points when	
$(s-2x)\tan\beta - \frac{x^2}{s}\tan^3\beta = 0$	
$s^2 - 2xs - x^2\tan^2\beta = 0$	**M1**
By quadratic formula: $s = x + x\sqrt{1+\tan^2\beta}$ (since $s > x$)	**M1**
Therefore $s = x(1+\sec\beta)$	**M1 A1 AG**
$s^2 = x^2(1+\sec\beta)^2$	
$a^2 - x^2\tan^2\beta = x^2(1+\sec\beta)^2$	**M1**
$x^2(1 + 2\sec\beta + \sec^2\beta + \tan^2\beta) = a^2$	
$2x^2\sec\beta\,(1+\sec\beta) = a^2$	**A1**
$A = x(x\sec\beta)\tan\beta$	**M1**
$A = \frac{1}{2}a^2\left(\frac{\tan\beta}{1+\sec\beta}\right)$	**A1**
$= \frac{1}{2}a^2\left(\frac{\sin\beta}{\cos\beta+1}\right)$	
$= \frac{1}{2}a^2\left(\frac{2\sin\frac{1}{2}\beta\cos\frac{1}{2}\beta}{2\cos^2\frac{1}{2}\beta - 1 + 1}\right)$	**M1**
$= \frac{1}{2}a^2\left(\tan\frac{1}{2}\beta\right)$	**A1 AG**
$\tan\angle ROS = \frac{x\tan\beta}{x(1+\sec\beta)}$	
$\tan\angle ROS = \tan\frac{1}{2}\beta$ so $\angle ROS = \frac{1}{2}\beta$	**B1 AG**

Question 5

B1	Clear explanation of the formula for the area of the rectangle.
B1	Coordinates of R deduced (use of Pythagoras / radius of circle).
M1	Equating y-coordinates for Q and R.
A1	Expression for s in terms of a, x and β.
M1	Differentiation to find $\frac{ds}{dx}$
M1	Differentiation of the expression for A.
A1	Correct differentiated expression.
M1	Substitution for $\frac{ds}{dx}$
A1	Rearrangement to required form.
ALTERNATIVE	
dM1	Substitution.
M1	Differentiation with respect to x (must either treat $\tan\beta$ as constant or have an expression involving $\frac{d\beta}{dx}$.)
A1	Correct derivative.
M1	Use of other formula.
A1	Correctly deduced relationship.
	Subtotal: 9
M1	Use of $\frac{ds}{dx} = 0$.
M1	Use of quadratic formula.
M1	Use of $1 + \tan^2\beta \equiv \sec^2\beta$
A1	Correct value for s determined, including justification of which root of the quadratic.
	Subtotal: 4
M1	Attempt to eliminate s from equation found in the first part of the question.
A1	Correct relationship.
M1	Substitution into the formula for the area.
A1	Correct formula obtained.
M1	Use of double angle formulae.
A1	Correct expression for the area.
B1	Justification for the size of angle ROS.
	Subtotal: 7

Question 6

(i) Suppose that $f(x) \geq 0$ for all values of x in the interval. Then $\int_0^1 f(x)\,dx > 0$ (since $f(x) \neq 0$ for some value of x in the interval). **B1**

Similarly, if $f(x) \leq 0$ for all values of x in the interval. Then $\int_0^1 f(x)\,dx < 0$ (since $f(x) \neq 0$ for some value of x in the interval).

Therefore if $\int_0^1 f(x)\,dx = 0$, $f(x)$ must take both positive and negative values in the interval $0 \leq x \leq 1$. **B1**

(ii) $\int_0^1 (x-\alpha)^2\, g(x)dx = \int_0^1 x^2\, g(x)dx - 2\alpha \int_0^1 x\, g(x)dx + \alpha^2 \int_0^1 g(x)\,dx$ **M1**

$= \alpha^2 - 2\alpha^2 + \alpha^2 = 0$ **A1**

Since $\int_0^1 g(x)\,dx = 1$, $g(x)$ must be non-zero for some value of x in the interval $0 \leq x \leq 1$. **M1**

Therefore, by **(i)**, $g(x)$ takes both positive and negative values in the interval $0 \leq x \leq 1$. **M1**

Therefore, $g(x)$ takes the value 0 at some point in the interval $0 \leq x \leq 1$. **A1**

If $g(x) = a + bx$, then:

$\int_0^1 g(x)dx = \left[ax + \frac{1}{2}bx^2\right]_0^1$

Therefore, $a + \frac{1}{2}b = 1$ **B1**

$\int_0^1 xg(x)dx = \left[\frac{1}{2}ax^2 + \frac{1}{3}bx^3\right]_0^1$

Therefore, $\frac{1}{2}a + \frac{1}{3}b = \alpha$ **B1**

$\int_0^1 x^2 g(x)dx = \left[\frac{1}{3}ax^3 + \frac{1}{4}bx^4\right]_0^1$

Therefore, $\frac{1}{3}a + \frac{1}{4}b = \alpha^2$ **B1**

From the first two equations:

$a = 4 - 6\alpha,\ b = 12\alpha - 6$

Substituting into the third equation:

$\frac{1}{3}(4 - 6\alpha) + \frac{1}{4}(12\alpha - 6) = \alpha^2$ **M1**

$\alpha^2 - \alpha + \frac{1}{6} = 0$

Therefore $\alpha = \frac{3 \pm \sqrt{3}}{6}$ **A1**

So $g(x) = 1 - \sqrt{3} + 2x\sqrt{3}$ or $g(x) = 1 + \sqrt{3} - 2x\sqrt{3}$ **A1**

Therefore the values of $g(0)$ and $g(1)$ are $1 - \sqrt{3}$ and $1 + \sqrt{3}$ and so $g(x) = 0$ for some value of x in the interval $0 \leq x \leq 1$. **E1**

(iii) $\int_0^1 h'(x)\,dx = h(1) - h(0) = 1$ **B1**

$\int_0^1 xh'(x)\,dx = [xh(x)]_0^1 - \int_0^1 h(x)\,dx$

So $\int_0^1 xh'(x)\,dx = 1 - \beta$ **M1 A1**

$\int_0^1 x^2 h'(x)\,dx = [x^2 h(x)]_0^1 - 2\int_0^1 xh(x)\,dx$

So $\int_0^1 x^2 h'(x)\,dx = 1 - \beta(2 - \beta) = (1 - \beta)^2$ **M1 A1**

Therefore, the conditions of (*) are met ($g(x) = h'(x)$, $\alpha = 1 - \beta$).

Therefore $h'(x) = 0$ for some value of x in the interval $0 \leq x \leq 1$. **B1**

Question 6

B1	Correct statement considering either $f(x)$ always positive or always negative.
B1	Statement that corresponding result is true in the other case and conclusion of proof by contradiction.
	Subtotal: 2
M1	Expansion of $(x-\alpha)^2$ and split of integral.
A1	Establish that the value of the integral is 0.
M1	Identify one of the conditions required to apply the result from **(i)**
M1	Apply result from **(i)**
A1	Draw the required conclusion.
	Subtotal: 5
B1	Relationship between a and b.
B1	Relationship between a, b and α.
B1	Relationship between a, b and α.
M1	Elimination of two of the variables.
A1	Value of one of the variables found.
A1	Correct choice of $g(x)$.
B1	Verification that there is a root in the interval.
	Subtotal: 7
B1	Confirm that first condition is satisfied.
M1	Use of integration by parts.
A1	Confirm that second condition is satisfied.
M1	Use of integration by parts.
A1	Confirm that the third condition is satisfied.
B1	Apply the previous result to draw the conclusion.
	Subtotal: 6

Question 7

(i)	CMA is an isosceles triangle, with $\|CA\| = b$ and $\|CM\| = \|AM\|$	
	Angle $CMA = 120°$	**B1**
	$b^2 = 2\|CM\|^2 - 2\|CM\|^2 \cos 120$	**M1**
	$\|CM\|^2 = \frac{b^2}{3}$	
	$\|CM\| = \frac{b}{\sqrt{3}}$	**A1 AG**
	$\|CL\| = \frac{a}{\sqrt{3}}$	**B1**
(ii)	$\|LM\|^2 = \|CM\|^2 + \|CL\|^2 - 2\|CM\|\|CL\| \cos \angle LCM$ $\|LM\|^2 = \frac{a^2}{3} + \frac{b^2}{3} - \frac{2ab}{3} \cos \angle LCM$	**M1**
	$\|AB\|^2 = \|BC\|^2 + \|CA\|^2 - 2\|BC\|\|CA\| \cos \angle ACB$ $c^2 = a^2 + b^2 - 2ab \cos \angle ACB$	**M1**
	$\angle LCM = \angle ACB + 60°$	**M1**
	Therefore $\cos \angle LCM = \frac{1}{2} \cos \angle ACB - \frac{\sqrt{3}}{2} \sin \angle ACB$	**M1**
	$\|LM\|^2 = \frac{a^2}{3} + \frac{b^2}{3} - \frac{ab}{3}(\cos \angle ACB - \sqrt{3} \sin \angle ACB)$	**M1**
	$\|LM\|^2 = \frac{a^2}{3} + \frac{b^2}{3} + \frac{1}{6}c^2 - \frac{1}{6}a^2 - \frac{1}{6}b^2 + \frac{ab\sqrt{3}}{3} \sin \angle ACB$	
	$6\|LM\|^2 = a^2 + b^2 + c^2 + 4\sqrt{3}\Delta$	**M1 A1 AG**
	A similar argument will show that $6\|MN\|^2 = 6\|NL\|^2 = a^2 + b^2 + c^2 + 4\sqrt{3}\Delta$, so LMN is an equilateral triangle.	**B1**
	The area of LMN is $\frac{\sqrt{3}}{4}\|LM\|^2$	
	If $a^2 + b^2 + c^2 = 4\sqrt{3}\Delta$, then the area of LMN is $\frac{\sqrt{3}}{4} \times \frac{4\sqrt{3}}{3}\Delta = \Delta$	**B1**
	If the area of LMN is equal to the area of ABC: $\frac{\sqrt{3}}{24}\left(a^2 + b^2 + c^2 + 4\sqrt{3}\Delta\right) = \Delta$ $\sqrt{3}(a^2 + b^2 + c^2) + 12\Delta = 24\Delta$	
	$a^2 + b^2 + c^2 = 4\sqrt{3}\Delta$	**B1**
(iii)	If $(a-b)^2 = -2ab(1 - \cos(C - 60°))$, then $a^2 - 2ab + b^2 = -2ab + 2ab\cos(C - 60°)$	
	$a^2 + b^2 = ab\cos C + ab\sqrt{3}\sin C$	**M1**
	$\Delta = \frac{1}{2}ab\sin C$, and by the cosine rule $c^2 = a^2 + b^2 - 2ab\cos C$	**M1**
	$a^2 + b^2 = \frac{1}{2}(a^2 + b^2 - c^2) + 2\sqrt{3}\Delta$	**A1**
	All steps are reversible, so the conditions are equivalent.	**B1**
	The areas of the triangles are equal if and only if $a^2 + b^2 + c^2 = 4\sqrt{3}\Delta$, which is equivalent to $(a-b)^2 = -2ab(1 - \cos(C - 60°))$.	
	Since $(a-b)^2 \geq 0$ and $-2ab(1 - \cos(C - 60°)) \leq 0$ this can only be satisfied if both sides are equal to 0.	**M1**
	Therefore $a = b$ and $\cos(C - 60°) = 1$, so $C = 60°$. So ABC is an equilateral triangle.	**A1**

Question 7

B1	Value of angle CMA.
M1	Use of cosine rule for triangle CMA.
A1	Correct value reached.
B1	Correct value stated.
	Subtotal: 4
M1	Application of cosine rule to triangle CLM.
M1	Application of cosine rule to triangle ABC.
M1	Relationship between angles LCM and ACB.
M1	Application of $\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$.
M1	Combination of the previous results.
M1	Use of $\Delta = \frac{1}{2} ab \sin C$
A1	Fully correct solution.
B1	Deduction that the triangle is equilateral.
B1	Justification that the condition implies that the areas are equal.
B1	Justification that the areas being equal implies that the condition holds.
	Subtotal: 10
M1	Expansion and rearrangement.
M1	Use of area of triangle and cosine rule.
A1	Fully correct justification.
B1	Clear indication that the reasoning applies both ways.
M1	Observation that inequality can only be satisfied in this case if both sides are 0.
A1	Clear explanation that this implies that the triangle is equilateral.
	Subtotal:6

Question 8

	Check $n = 1$: $a_1^2 + 2a_1b_1 - b_1^2 = 1^2 + 2(1)(2) - 2^2 = 1$	**B1**
	Assume that the result is true for $n = k$: $$a_k^2 + 2a_kb_k - b_k^2 = 1$$	
	$a_{k+1}^2 + 2a_{k+1}b_{k+1} - b_{k+1}^2$ $= (a_k + 2b_k)^2 + 2(a_k + 2b_k)(2a_k + 5b_k) - (2a_k + 5b_k)^2$	**M1 A1**
	$= a_k^2 + 2a_kb_k - b_k^2 = 1$	
	Therefore, by induction, $a_n^2 + 2a_nb_n - b_n^2 = 1$ for all $n \geq 1$	**B1**
(i)	From the definitions of the sequences $a_n > 0$ and $b_n > 0$ for all n	**B1**
	$b_1 = 2 \geq 2 \times 5^{1-1}$, so $b_n \geq 2 \times 5^{n-1}$ is true in the case $n = 1$.	**B1**
	Assume that $b_k \geq 2 \times 5^{k-1}$ for some value k.	
	Then $b_{k+1} = 2a_k + 5b_k \geq 2 \times 5^k$	**M1 A1**
	Therefore , by induction, $b_n \geq 2 \times 5^{n-1}$ for all $n \geq 1$	**B1**
	ALTERNATIVE	
	$a_n \geq 0$ for all $x \geq 1$	**B1**
	$b_1 = 2$, $b_n = 2a_{n-1} + 5b_{n-1} \geq 5b_{n-1} \geq 5^2b_{n-2} \geq \cdots \geq 5^{n-1}b_1$	**M1 A1**
	$b_n \geq 2(5^{n-1})$	**B2**
	From (*): $c_n^2 + 2c_n - 1 = \left(\frac{1}{b_n}\right)^2$	**B1**
	Therefore, as $n \to \infty$, c_n approaches a root of $x^2 + 2x - 1 = 0$	**M1**
	The roots of $x^2 + 2x - 1 = 0$ are $-1 \pm \sqrt{2}$	**A1**
	Since $c_n > 0$, $c_n \to \sqrt{2} - 1$ as $n \to \infty$.	**A1**
(ii)	$c_{n+1} = \frac{a_{n+1}}{b_{n+1}} = \frac{a_n+2b_n}{2a_n+5b_n}$	**M1**
	Therefore $c_{n+1} - c_n = \frac{a_n+2b_n}{2a_n+5b_n} - \frac{a_n}{b_n} = \frac{-2a_n^2-4a_nb_n+2b_n^2}{(2a_n+5b_n)b_n}$	
	$c_{n+1} - c_n = \frac{-2}{(2a_n+5b_n)b_n} < 0$	**M1**
	Therefore the sequence c_n is decreasing.	**A1**
	Therefore $c_n > \sqrt{2} - 1$.	**A1 AG**
	$c_n + 1 > \sqrt{2}$ and so $\frac{2}{c_n+1} < \sqrt{2} < c_n + 1$	**A1 AG**
	ALTERNATIVE FOR $c_n > \sqrt{2} - 1$	
	$c_n = -1 + \sqrt{2 + \frac{1}{b_n^2}}$	**M1 A1**
	$b_n > 0$	**B1**
	$c_n > \sqrt{2} - 1$	**A1**
	$a_1 = 1$ and $b_1 = 2$, so $c_1 = \frac{1}{2}$	
	$a_2 = 5$ and $b_2 = 12$, so $c_2 = \frac{5}{12}$	
	$a_3 = 29$ and $b_3 = 70$, so $c_3 = \frac{29}{70}$	**M1**
	Therefore $\frac{2}{\frac{29}{70}+1} < \sqrt{2} < \frac{29}{70} + 1$	
	$\frac{140}{99} < \sqrt{2} < \frac{99}{70}$	**A1**

Question 8

B1	Check the case $n = 1$
M1	Attempt to relate the case $n = k + 1$ to the case $n = k$
A1	Deduce that the result holds in the case $n = k + 1$ if it holds in the case $n = k$
B1	Conclusion of proof by induction.
	Subtotal: 4
B1	Observe that all values in both sequences are positive.
B1	Check the case $n = 1$
M1	Attempt to relate the case $n = k + 1$ to the case $n = k$
A1	Deduce that the result holds in the case $n = k + 1$ if it holds in the case $n = k$
B1	Conclusion of proof by induction.
ALTERNATIVE	
B1	Observe that all values of a_n are positive.
M1	Inequality between consecutive values for b_n
A1	Repeated application of inequality.
B2	Conclusion clearly justified.
	Subtotal: 5
B1	Deduce formula satisfied by c_n
M1	Find equation satisfied by limit of sequence.
A1	Solve quadratic.
A1	Justify choice of positive root.
	Subtotal: 4
M1	Write c_{n+1} in terms of a_n and b_n
M1	Find expression for $c_{n+1} - c_n$
A1	Conclude that the sequence is decreasing.
A1	Explain why this shows that $c_n > \sqrt{2} - 1$
ALTERNATIVE	
M1	Solution of quadratic.
A1	Choice of positive square root.
A1	Observe that $b_n > 0$
A1	Clear explanation that $c_n > \sqrt{2} - 1$
A1	Conclude required inequality
M1	Calculate c_3
A1	Deduce required inequality.
	Subtotal: 7

Question 9

(i) Horizontal speed = $u\cos\alpha$, therefore the particle passes through P after $\frac{d}{u\cos\alpha}$ seconds. **M1 A1**

Vertically:
Initial speed = $u\sin\alpha$, acceleration $= -g$, displacement $= d\tan\beta$. **M1 A1**

$d\tan\beta = u\sin\alpha\left(\frac{d}{u\cos\alpha}\right) - \frac{1}{2}g\left(\frac{d}{u\cos\alpha}\right)^2$

$d\tan\beta = d\tan\alpha - \frac{d^2 g}{2u^2}\sec^2\alpha$

$u^2 = \frac{dg\sec^2\alpha}{2(\tan\alpha - \tan\beta)}$

u will be as small as possible at a point where $\frac{du}{d\alpha} = 0$:

$2u\frac{du}{d\alpha} = \frac{2(\tan\alpha-\tan\beta)(2dg\sec^2\alpha\tan\alpha) - dg\sec^2\alpha(2\sec^2\alpha)}{4(\tan\alpha-\tan\beta)^2}$ **M1 M1 A1**

$2u\frac{du}{d\alpha} = \frac{2dg\sec^2\alpha((\tan\alpha-\tan\beta)(2\tan\alpha)-\sec^2\alpha)}{4(\tan\alpha-\tan\beta)^2}$

Therefore $\frac{du}{d\alpha} = 0$ if $(\tan\alpha - \tan\beta)(2\tan\alpha) - \sec^2\alpha = 0$

$(\tan\alpha - \tan\beta)(2\tan\alpha) - \tan^2\alpha - 1 = 0$ **M1**

$\tan^2\alpha - 2\tan\alpha\tan\beta - 1 = 0$

So $\tan\beta = \frac{\tan^2\alpha - 1}{2\tan\alpha}$ and $\tan\alpha - \tan\beta = \frac{\tan^2\alpha+1}{2\tan\alpha} = \frac{\sec^2\alpha}{2\tan\alpha}$ **A1**

Therefore $u^2 = \frac{dg\sec^2\alpha}{2\left(\frac{\sec^2\alpha}{2\tan\alpha}\right)} = dg\tan\alpha$ **M1 A1 AG**

$\tan\beta = \frac{-1}{\tan 2\alpha} = -\cot 2\alpha$ **M1**

The graph of $y = -\cot x$ is a translation of the graph of $y = \tan x$ by $90(°)$ horizontally. **M1 A1 AG**

α must be greater than β.

Therefore $2\alpha = \beta + 90°$

(ii) When the particle passes through P:

Horizontal velocity is $u\cos\alpha$

Vertical velocity is $u\sin\alpha - \frac{gd}{u\cos\alpha}$ **B1**

Therefore if the angle to the horizontal is γ, then

$\tan\gamma = \frac{u\sin\alpha - \frac{gd}{u\cos\alpha}}{u\cos\alpha} = \tan\alpha - \frac{gd}{u^2}\sec^2\alpha$ **M1 A1**

Since $u^2 = gd\tan\alpha$: $\tan\alpha\tan\gamma = \tan^2\alpha - \sec^2\alpha = -1$ **M1 A1**

Therefore $\gamma = \alpha - 90°$ (or $(90-\alpha)°$ below the horizontal). **A1**

Question 9

M1	Use of constant horizontal velocity to determine time passing through P.
A1	Correct expression for time.
M1	Uniform acceleration formula.
A1	Correct equation.
M1	Attempt to differentiate either u or u^2 with respect to α
M1	Application of quotient rule.
A1	Correctly differentiated.
M1	Set derivative equal to 0.
A1	Rearrange to get formula for $\tan\beta$
M1	Substitute to get expression for u^2
A1	Fully correct justification.
M1	Observe relationship between $\tan\beta$ and $\cot 2\alpha$
M1	Reference to the relationship between the two functions.
A1	Correct relationship, fully justified.
	Subtotal: 14
B1	Calculation of vertical velocity through P (must be in terms of α)
M1	Division of two velocities to get tan of required angle.
A1	Simplified form.
M1	Substitution of result from part (i)
A1	$\tan\alpha\tan\gamma = -1$
A1	Removal of $\tan$ functions.
	Subtotal: 6

Question 10

(i)	Conservation of momentum: $mu = mv + \lambda m u_1$	**B1**
	Law of Restitution: $u_1 - v = eu$	**B1**
	Eliminating v: $u_1 - (u - \lambda u_1) = eu$ $u_1(1+\lambda) = (1+e)u$	**M1**
	$u_1 = \frac{1+e}{1+\lambda}u$	**A1 AG**
	For the first collision with particle n $(n > 1)$: $\lambda^{n-1} m u_{n-1} = \lambda^{n-1} m v_{n-1} + \lambda^n m u_n$ $u_{n-1} = v_{n-1} + \lambda u_n$ $u_n - v_{n-1} = e u_{n-1}$	**M1 A1**
	Eliminating v_{n-1}: $u_{n-1} = u_n - e u_{n-1} + \lambda u_n$ $(1+e)u_{n-1} = u_n(1+\lambda)$	
	Therefore $u_n = \left(\frac{1+e}{1+\lambda}\right)^n u$	**A1**
	$v_{n-1} = \left(\frac{1+e}{1+\lambda}\right)^{n-1} u - \lambda \left(\frac{1+e}{1+\lambda}\right)^n u$	**M1**
	$= \left(\frac{1+e}{1+\lambda}\right)^{n-1} u \left(1 - \frac{\lambda(1+e)}{1+\lambda}\right)$ $= \left(\frac{1-e\lambda}{1+\lambda}\right)\left(\frac{1+e}{1+\lambda}\right)^{n-1} u$	**M1**
	So $v_n = \left(\frac{1-e\lambda}{1+\lambda}\right)\left(\frac{1+e}{1+\lambda}\right)^n u$	**A1**
(ii)	If $e > \lambda$ then $\left(\frac{1+e}{1+\lambda}\right) > 1$, so $v_{k+1} > v_k$ for every choice of k and so there cannot be any subsequent collisions.	**M1 M1** **A1**
(iii)	If $e = \lambda$ then all particles will have the same velocity after their second collision. $v_n = (1-e)u$	**M1 A1**
	The Kinetic Energies of the particles after their second collisions will form a geometric series with first term $\frac{1}{2}m(1-e)^2u^2$ and common ratio e.	**M1**
	Therefore the sum will approach $\frac{\frac{1}{2}m(1-e)^2u^2}{1-e} = \frac{1}{2}m(1-e)u^2$	**A1**
	The initial KE was $\frac{1}{2}mu^2$, so the fraction that has been lost approaches e.	**A1 AG**
(iv)	If $\lambda e = 1$ then all particles stop after their second collision.	**B1**
	All of the energy is lost eventually in this case.	**B1**

Question 10

B1	Correct equation.
B1	Correct equation.
M1	Attempt to eliminate v
A1	Reach given equation correctly.
M1	Consideration of conservation of momentum for n^{th} collision.
A1	Simplified form.
A1	Correct equation for u_n
M1	Substitution to find v_{n-1}
M1	Simplification.
A1	Adjustment to get v_n
	Subtotal: 10
M1	$\left(\frac{1+e}{1+\lambda}\right) > 1$
M1	Relationship between velocities.
A1	Clear explanation why this implies no further collisions.
	Subtotal: 3
M1	Comment that all velocities will be equal.
A1	Correct common velocity stated.
M1	Identify that the KEs will form a geometric series.
A1	Sum to infinity.
A1	Clear justification that fraction of KE lost approaches e
	Subtotal: 5
B1	Observation that all particles stop.
B1	All KE lost (fraction lost = 1)
	Subtotal: 2

Question 11

Forces at A:	
Reaction force R_A (perpendicular to the slope)	**B1**
Frictional force F_A (parallel to slope, towards O)	
Forces at B:	
Reaction force R_B (parallel to slope)	**B1**
Frictional force F_B (perpendicular to slope, away from O)	
Since equilibrium is limiting at both A and B: $F_A = R_A \tan\gamma$ and $F_B = R_B \tan\gamma$	**B1**
Resolving parallel to the slope: $R_A \tan\gamma + W\sin\alpha = R_B$	**M1 A1**
Resolving perpendicular to the slope: $W\cos\alpha = R_A + R_B \tan\gamma$	**M1 A1**
Eliminating W:	**M1**
$W\sin\alpha\cos\alpha = R_B\cos\alpha - R_A\tan\gamma\cos\alpha = R_A\sin\alpha + R_B\tan\gamma\sin\alpha$	**M1**
$R_A\tan\alpha + R_B\tan\alpha\tan\gamma = R_B - R_A\tan\gamma$	**M1**
$R_A = \frac{1-\tan\alpha\tan\gamma}{\tan\alpha+\tan\gamma} R_B$, so $R_B = \tan(\alpha+\gamma) R_A$	**A1**
Taking moments about the centre of the rod:	**M1 M1**
$L\cos\beta\, R_B + L\sin\beta\, R_B\tan\gamma + L\cos\beta\, R_A\tan\gamma = L\sin\beta\, R_A$	**A1**
So $R_B + R_B\tan\beta\tan\gamma + R_A\tan\gamma = R_A\tan\beta$	**M1 M1**
Therefore: $\tan(\alpha+\gamma) + \tan(\alpha+\gamma)\tan\beta\tan\gamma + \tan\gamma = \tan\beta$	**M1**
$\tan\beta = \frac{\tan(\alpha+\gamma)+\tan\gamma}{1-\tan(\alpha+\gamma)\tan\gamma} = \tan(\alpha+2\gamma)$	**M1**
Therefore $\beta = \alpha + 2\gamma + n\pi$	**M1**
Since $\alpha < \beta$ and β is acute, $\beta = \alpha + 2\gamma$.	**A1 AG**

Question 11

B1	Identification of the forces at A (may be implied by later work).
B1	Identification of the forces at B (may be implied by later work).
B1	Use of limiting equilibrium at both points.
M1	Resolve parallel to slope.
A1	All correct.
M1	Resolve perpendicular to slope.
A1	All correct.
M1	Elimination of any one variable from equations.
M1	Manipulation of trigonometric functions (may occur later in solution).
M1	Use of $\tan(A+B)$ (or equivalent) formula (may occur later in solution).
A1	Correct relationship between two reaction forces.
M1	Take moments about centre of rod (at least 2 correct)
M1	Moments about centre of rod (at least 3 correct)
A1	Fully correct.
M1	Cancel L from the equation.
M1	Apply $\tan\theta = \frac{\sin\theta}{\cos\theta}$
M1	Eliminate so that W, R_A and R_B are not present in the equation.
M1	Rearrange to apply $\tan(A+B)$ formula.
M1	Full solutions to tan equation just reached.
A1	Deduce relationship between α, β and γ, explaining why it can’t be any of the others.
	Subtotal: 20

Question 12

(i) The probability that any one participant will choose the correct number is $\frac{1}{N}$.

Therefore, $P(No\ participant\ picks\ the\ winning\ ticket) = \left(1 - \frac{1}{N}\right)^N$ **M1 A1**

The expected amount that will need to be paid in prizes is $\left(1 - \left(1 - \frac{1}{N}\right)^N\right)J$. **M1**

Therefore the expected profit is $Nc - \left(1 - \left(1 - \frac{1}{N}\right)^N\right)J$. **A1**

Therefore the expected profit is approximately $Nc - \left(1 - \frac{1}{e}\right)J$ **A1**

If $2Nc = J$ then the expected profit is $\left(\frac{1}{2} - 1 + \frac{1}{e}\right)J < 0$, therefore the organizer will expect to make a loss. **A1 AG**

(ii) The probability of picking a number between 1 and N is

$\gamma N \times \frac{a}{N} + (1-\gamma)N \times \frac{b}{N} = 1$ **M1**

$a\gamma + b - b\gamma = 1$ **A1**

If the number that is drawn is popular then the probability that no participant will choose it is $\left(1 - \frac{a}{N}\right)^N$

If the number that is drawn is not popular then the probability that no participant will choose it is $\left(1 - \frac{b}{N}\right)^N$ **B1**

The probability that no participant chooses the winning number is therefore:

$\gamma\left(1 - \frac{a}{N}\right)^N + (1-\gamma)\left(1 - \frac{b}{N}\right)^N$ **M1 A1**

The expected profit is therefore

$Nc - \left(1 - \gamma\left(1 - \frac{a}{N}\right)^N - (1-\gamma)\left(1 - \frac{b}{N}\right)^N\right)J$ **A1**

which can be approximated to

$Nc - \left(1 - \gamma e^{-a} - (1-\gamma)e^{-b}\right)J = \gamma J e^{-a} + (1-\gamma)Je^{-b} + Nc - J$ **M1 A1**

If $\gamma = \frac{1}{8}$, then $\frac{a}{8} + \frac{7b}{8} = 1$. If $a = 9b$, then $b = \frac{1}{2}$ **M1**

$a = \frac{9}{2}$ **A1**

If $2Nc = J$, then the profit will be:

$\frac{Nc}{4}e^{-\frac{9}{2}} + \frac{7Nc}{4}e^{-\frac{1}{2}} - Nc = \frac{Nc}{4}\left(e^{-\frac{9}{2}} + 7e^{-\frac{1}{2}} - 4\right)$ **M1**

$e^{-\frac{9}{2}} + 7e^{-\frac{1}{2}} - 4 = e^{-\frac{1}{2}}(e^{-4} + 7) - 4 > \frac{7\sqrt{3}}{3} - 4$, since $e < 3$ **M1**

$\left(\frac{7\sqrt{3}}{3}\right)^2 = \frac{147}{9} > 16$, so $\frac{7\sqrt{3}}{3} - 4 > 0$, meaning that the organiser will expect to make a profit. **M1 A1**

Question 12

M1	Identification of the probability of choosing the winning number.
A1	Correct probability that no participant chooses the winning number.
M1	Expected amount to be paid out.
A1	Correct expected profit.
A1	Use of approximation.
A1	Justification that organizer will expect to make a loss.
	Subtotal: 6
M1	Consideration of probability that the number chosen is between 1 and N.
A1	Correct relationship.
B1	Correct probabilities of no winner for both cases.
M1	Find probability that no participant chooses a winning ticket.
A1	Correct probability.
A1	Correct expected profit.
M1	Use of approximation.
A1	Simplification to required form.
	Subtotal: 8
M1	Substitution and attempt to solve simultaneous equations.
A1	Values of a and b correct.
M1	Profit calculated in the case $2Nc = J$
M1	Rearranged and use of $e < 3$
M1	Attempt to show that the expected profit is positive.
A1	Fully clear explanation.
	Subtotal: 6

Question 13

$s_1 = 0$	**B1**
If the r^{th} slice is to be used to make toast then either the $(r-1)^{th}$ slice was used as the second slice for a sandwich (probability s_{r-1}) the $(r-1)^{th}$ slice was used for toast (probability t_{r-1})	**M1**
The probability that the next slice is used for toast is p. Therefore $t_r = (s_{r-1} + t_{r-1})p$	**A1**
The r^{th} slice being the second slice for a sandwich is equivalent to the $(r-1)^{th}$ slice being the first slice for a sandwich, so the probability that the $(r-1)^{th}$ slice is the first slice for a sandwich is also s_r	**M1 M1**
Since there are only three possibilities for the use of a slice, $s_{r-1} + t_{r-1} + s_r = 1$ and so $s_r = 1 - (s_{r-1} + t_{r-1})$	**A1**
Valid for $r \geq 2$ as the reasoning only refers to the previous slice. Formula for t_r is not valid for $r = n$ as the final slice must be toast.	**B1**
$s_{r-1} + t_{r-1} = 1 - s_r$, so $t_r = p(1 - s_r)$	**M1**
Therefore $s_r = 1 - \left(s_{r-1} + p(1 - s_{r-1})\right)$	**M1**
$s_r = 1 - s_{r-1} - p(1 - s_{r-1}) = (1-p)(1 - s_{r-1})$	
$s_r = q(1 - s_{r-1})$	**A1**
$s_1 = \frac{q+(-q)}{1+q} = 0$, which is correct.	**B1**
Assume that $s_k = \frac{q+(-q)^k}{1+q}$	
Then $s_{k+1} = q\left(1 - \frac{q+(-q)^k}{1+q}\right)$	**M1**
$s_{k+1} = q\left(\frac{1+q-q-(-q)^k}{1+q}\right) = \frac{q+(-q)^{k+1}}{1+q}$	**A1**
Therefore, by induction, $s_r = \frac{q+(-q)^r}{1+q}$ for $1 \leq r \leq n-1$	**B1**
$ps_r = p - t_r$	**M1**
Therefore $t_r = p - p\left(\frac{q+(-q)^r}{1+q}\right)$ for $1 \leq r \leq n-1$	**A1**
$s_n = 1 - \left(\frac{q+(-q)^{n-1}}{1+q} + p - p\left(\frac{q+(-q)^{n-1}}{1+q}\right)\right)$	**M1**
$s_n = 1 - p - (1-p)\left(\frac{q+(-q)^{n-1}}{1+q}\right)$	
$s_n = \frac{q(1+q) - q\left(q+(-q)^{n-1}\right)}{1+q} = \frac{q+(-q)^n}{1+q}$	**A1**
Since the last slice must either be the second slice of a sandwich or toast: $t_n = 1 - \frac{q+(-q)^n}{1+q} = \frac{1-(-q)^n}{1+q}$	**M1 A1**

Question 13

B1	Correct value.
M1	Identification of the two possibilities.
A1	Clear justification of the equation.
M1	Identification of the probability that it is the first slice of a sandwich.
M1	Identification of the three possibilities in general.
A1	Clear justification of the equation.
B1	Clear justification of the ranges for which the equations are valid.
	Subtotal: 7
M1	Rearrangement.
M1	Substitution.
A1	Correct equation.
B1	Check first case.
M1	Relate case $k+1$ to case k.
A1	Show that the correct formula follows.
B1	Complete proof by induction.
M1	Use relationship between s and t.
A1	Correct equation, including range for which it is valid.
	Subtotal: 9
M1	Substitute into formula for s_n
A1	Correct formula.
M1	Observe that $t_n = 1 - s_n$
A1	Correct formula.
	Subtotal: 4

Question 1

(i) $I_n = \int_0^1 \arctan x \,.\, x^n \, dx$ — **M1** Use of intgrn. by parts (parts correct way round)

$= \left[\arctan x \,.\, \frac{x^{n+1}}{n+1} \right]_0^1 - \int_0^1 \frac{1}{1+x^2} . \frac{x^{n+1}}{n+1} \, dx$ — **A1** Correct to here

$= \left(\frac{\pi}{4} . \frac{1}{n+1} - 0 \right) - \frac{1}{n+1} \int_0^1 \frac{x^{n+1}}{1+x^2} \, dx$

$\Rightarrow (n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2} \, dx$ — **A1** Given Answer legitimately established **3**

Setting $n = 0$, $I_0 = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} \, dx$ — **M1** Attempt to solve this using recognition/ substitution

$= \frac{\pi}{4} - \left[\tfrac{1}{2} \ln \left(1 + x^2 \right) \right]$ — **M1** Log integral involved

$= \frac{\pi}{4} - \tfrac{1}{2} \ln 2$ — **A1 CAO** **3**

(ii) $n \to n + 2$ in given result:

$(n+3)I_{n+2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2} \, dx$ — **B1** Noted or used somewhere

$(n+3)I_{n+2} + (n+1)I_n = \frac{\pi}{2} - \int_0^1 \frac{x^{n+1}\left(1+x^2\right)}{1+x^2} \, dx$ — **M1** Adding and cancelling ready to integrate

$= \frac{\pi}{2} - \frac{1}{n+2}$ — **A1 CAO** **3**

Setting $n = 0$ and then $n = 2$ in this result (or equivalent involving integrals):

$3I_2 + I_0 = \frac{\pi}{2} - \frac{1}{2}$ and $5I_4 + 3I_2 = \frac{\pi}{2} - \frac{1}{4}$ — **M1**

Eliminating I_2 and using value for I_0 to find I_4 — **M1** By subtracting, or equivalent

$I_4 = \frac{1}{20}\left(1 + \pi - 2\ln 2\right)$ — **A1 FT** from their I_0 value **3**

(iii) For $n = 1$, $5I_4 = A - \tfrac{1}{2}\left(-1 + \tfrac{1}{2}\right) = A + \tfrac{1}{4}$

$= \tfrac{1}{4} + \tfrac{1}{4}\pi - \tfrac{1}{2}\ln 2$ — **M1** Comparing formula with found I_4 value

and the result is true for $n = 1$ provided

$A = \tfrac{1}{4}\pi - \tfrac{1}{2}\ln 2$ — **A1 FT** from their I_4 value **2**

Assuming $(4k+1)I_{4k+1} = A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r \frac{1}{r}$ **M1** For a clearly stated induction hypothesis

(or a fully explained "if … then …" at end)

$(4k+5)I_{4k+4} + (4k+3)I_{4k+2} = \frac{\pi}{2} - \frac{1}{4k+4}$ **B1**

$(4k+3)I_{4k+2} + (4k+1)I_{4k} = \frac{\pi}{2} - \frac{1}{4k+2}$ **B1**

Subtracting:

$(4k+5)I_{4k+4} = (4k+1)I_{4k} + \frac{1}{4k+2} - \frac{1}{4k+4}$ **M1**

$= A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r \frac{1}{r} + \frac{1}{4k+2} - \frac{1}{4k+4}$ **M1** Use of assumed result

$= A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r \frac{1}{r} - \frac{1}{2}(-1)^{2k+1}\frac{1}{2k+1} - \frac{1}{2}(-1)^{2k+2}\frac{1}{2k+2}$

$= A - \frac{1}{2}\sum_{r=1}^{2(k+1)}(-1)^r \frac{1}{r}$ **A1** A clear demonstration of how the two extra terms fit must be given

6

Question 2

Let $x_n = X$. Then $x_{n+1} = \dfrac{aX-1}{X+b}$ and $x_{n+2} = \dfrac{a\left(\dfrac{aX-1}{X+b}\right)-1}{\left(\dfrac{aX-1}{X+b}\right)+b}$ **M1 A1** Correct, unsimplified

i.e. $x_{n+2} = \dfrac{(a^2-1)X-(a+b)}{(a+b)X+(b^2-1)}$ **M1** Attempt to remove "fractions within fractions"

A1 Correct, simplified **4**

(i) If $x_{n+1} = x_n$ then $aX - 1 = X^2 + bX$ **M1**

$\Rightarrow\ 0 = X^2 - (a-b)X + 1$ **A1**

If $x_{n+2} = x_n$ then

$(a^2-1)X - (a+b) = (a+b)X^2 + (b^2-1)X$ **M1**

$\Rightarrow\ 0 = (a+b)\{X^2 - (a-b)X + 1\}$ **M1 A1** Factorisation

and so, for $x_{n+2} = x_n$ but $x_{n+1} \neq x_n$

we must have $a + b = 0$ **A1** **Given Answer** fully justified & clearly stated

(No marks for setting $b = -a$, for instance, and showing sufficiency)

For "comparing coefficients" approach (must be all 3 terms) max. 3/4 **6**

(ii) $x_{n+4} = \dfrac{(a^2-1)x_{n+2}-(a+b)}{(a+b)x_{n+2}+(b^2-1)}$ **M1** Use of the two-step result from earlier

$$= \frac{(a^2-1)\left[\dfrac{(a^2-1)X-(a+b)}{(a+b)X+(b^2-1)}\right]-(a+b)}{(a+b)\left[\dfrac{(a^2-1)X-(a+b)}{(a+b)X+(b^2-1)}\right]+(b^2-1)}$$ **A1** Correct, unsimplified, in terms of X

If $x_{n+4} = x_n$ then **M1** Equating

$(a^2-1)^2X - (a+b)(a^2-1) - (a+b)^2X - (a+b)(b^2-1)$ **A1** LHS correct

$= (a+b)(a^2-1)X^2 - (a+b)^2X + (a+b)(b^2-1)X^2 + (b^2-1)^2X$ **A1** RHS correct

$\Rightarrow\ 0 = (a+b)(a^2+b^2-2)X^2 - [(a^2-1)^2 - (b^2-1)^2]X + (a+b)(a^2+b^2-2)$

M1 Good attempt to simplify

$\Rightarrow\ 0 = (a+b)(a^2+b^2-2)\{X^2 - (a-b)X + 1\}$ **M1** Factorisation attempt

A1 A1 Partial; complete

and the sequence has period 4 if and only if

$a^2 + b^2 = 2,\ a + b \neq 0,\ X^2 - (a-b)X + 1 \neq 0$ **B1 CAO** Correct final statement

[Ignore any discussion or confusion regarding issues of necessity and sufficiency]

NB Some candidates may use the one-step result repeatedly and get to x_{n+4} via x_{n+3}:

$x_{n+3} = \dfrac{(a^3-2a-b)X-(a^2+ab+b^2-1)}{(a^2++ab+b^2-1)X-(a+2b+b^3)}$ and $x_{n+4} = \dfrac{ax_{n+3}-1}{x_{n+3}+b}$ starts the process; then as above.

10

ALT. Consider the two-step sequence $\{\ldots, x_n, x_{n+2}, x_{n+4}, \ldots\}$ given by (assuming $a + b \neq 0$)

$$x_{n+2} = \frac{\left(\dfrac{a^2-1}{a+b}\right)X - 1}{X + \left(\dfrac{b^2-1}{a+b}\right)} \equiv \frac{AX-1}{X+B},$$ which is clearly of exactly the same form as before.

Then $x_{n+4} = x_n$ if and only if $a + b \neq 0$, $X^2 - (a-b)X + 1 \neq 0$ (from $x_{n+4} \neq x_{n+2}$ and $x_{n+4} \neq x_n$ as before), together with the condition $A + B = 0$ (also from previous work); i.e. $\dfrac{a^2-1}{a+b} + \dfrac{b^2-1}{a+b} = 0$, which is equivalent to $a^2 + b^2 - 2 = 0$ since $a + b \neq 0$.

Note that it is not necessary to consider $x_{n+4} \neq x_{n+3}$ since if $x_{n+4} = x_{n+3} = X$ then the sequence would be constant.

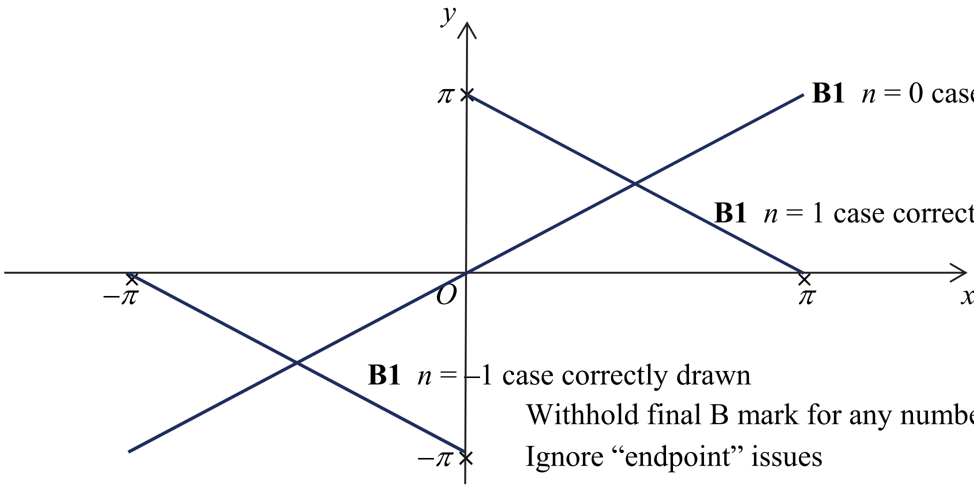
Question 3

(i) $\sin y = \sin x \Rightarrow y = n\pi + (-1)^n x$

$n = -1$: $y = -\pi - x$ **B1**

$n = 0$: $y = x$ **B1**

$n = 1$: $y = \pi - x$ **B1** Withhold final B mark for any number of extra eqns.



B1 $n = 0$ case correctly drawn

B1 $n = 1$ case correctly drawn

B1 $n = -1$ case correctly drawn

Withhold final B mark for any number of extra lines

Ignore "endpoint" issues

6

(ii) $\sin y = \frac{1}{2}\sin x \Rightarrow \cos y \dfrac{dy}{dx} = \frac{1}{2}\cos x$ **M1** Implicit diffn. attempt (or equivalent)

$\dfrac{dy}{dx} = \dfrac{\cos x}{2\cos y}$ **A1** Correct

$= \dfrac{\cos x}{2\sqrt{1 - \frac{1}{4}\sin^2 x}}$ or $\dfrac{\cos x}{\sqrt{4 - \sin^2 x}}$ **A1** Correct and in terms of x only **3**

$$\frac{d^2 y}{dx^2} = \frac{(4 - \sin^2 x)^{\frac{1}{2}}.-\sin x - \cos x.\frac{1}{2}(4 - \sin^2 x)^{-\frac{1}{2}}.-2\sin x\cos x}{4 - \sin^2 x}$$

M1 For use of the *Quotient Rule* (or equivalent)

M1 For use of the *Chain Rule* for d/dx(denominator)

A1

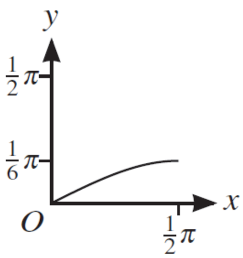
$$= \frac{-\sin x(4 - \sin^2 x) + \cos^2 x.\sin x}{(4 - \sin^2 x)^{\frac{3}{2}}}$$

M1 Method for getting correct denominator

$$= \frac{\sin x\{\cos^2 x - 4 + \sin^2 x\}}{(4 - \sin^2 x)^{\frac{3}{2}}}$$

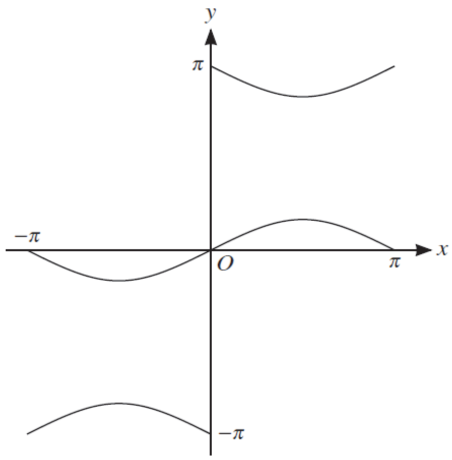
$$= \frac{-3\sin x}{(4 - \sin^2 x)^{\frac{3}{2}}}$$

A1 **Given Answer** correctly obtained from $c^2 + s^2 = 1$ **5**



Initially, $\frac{dy}{dx} = \frac{1}{2}$ at (0, 0) increasing to a maximum at $(\frac{\pi}{2}, \frac{\pi}{6})$ since $\frac{d^2y}{dx^2} < 0$

B1 (Gradient and coordinate details unimportant unless graphs look silly as a result)



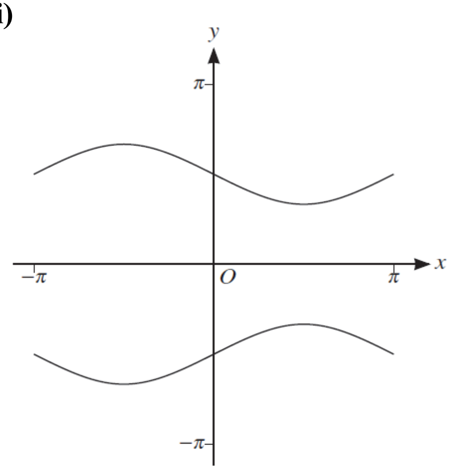
B1 Reflection symmetry in $x = \frac{\pi}{2}$

B1 Rotational symmetry about O

B1 Reflection symmetry in $y = \pm\ \frac{\pi}{2}$

4

(iii)



B1 RHS correct

B1 LHS correct

2

Question 4

(i) Setting f(x) = 1 in **(*)** gives

$$\left(\int_a^b g(x)\,dx\right)^2 \le \left(\int_a^b 1\,dx\right)\left(\int_a^b [g(x)]^2\,dx\right)$$ **B1** Clearly stated

Let g(x) = e^x : $\left(\int_a^b e^x\,dx\right)^2 \le (b-a)\left(\int_a^b e^{2x}\,dx\right)$ **M1**

$\Rightarrow \left(e^b - e^a\right)^2 \le (b-a).\tfrac{1}{2}\left(e^{2b} - e^{2a}\right)$

$\Rightarrow \left(e^b - e^a\right)^2 \le (b-a).\tfrac{1}{2}\left(e^b - e^a\right)\left(e^b + e^a\right)$

$\Rightarrow e^b - e^a \le \tfrac{1}{2}(b-a)\left(e^b + e^a\right)$ **A1**

Choosing a = 0 and b = t gives **M1**

$e^t - 1 \le \tfrac{1}{2}t\left(e^t + 1\right) \Rightarrow \dfrac{e^t - 1}{e^t + 1} \le \tfrac{1}{2}t$ **A1** **Given Answer** legitimately obtained **5**

(ii) Setting f(x) = x, a = 0 and b = 1 in **(*)** gives

$$\left(\int_0^1 x\,g(x)\,dx\right)^2 \le \left(\int_0^1 x^2\,dx\right)\left(\int_0^1 [g(x)]^2\,dx\right)$$ **B1** Clearly stated

Choosing g(x) = $e^{-\frac{1}{4}x^2}$ gives **M1**

$$\left(\int_0^1 xe^{-\frac{1}{4}x^2}\,dx\right)^2 \le \tfrac{1}{3}\left(1^3 - 0^3\right)\left(\int_0^1 e^{-\frac{1}{2}x^2}\,dx\right)$$

$$\left(\left[-2e^{-\frac{1}{4}x^2}\right]_0^1\right)^2 \le \tfrac{1}{3}\left(\int_0^1 e^{-\frac{1}{2}x^2}\,dx\right)$$ **A1 A1** LHS, RHS correct

$$\Rightarrow \int_0^1 e^{-\frac{1}{2}x^2}\,dx \ge 3\left(-2\left[-e^{-\frac{1}{4}} + 1\right]\right)^2$$

i.e. $\displaystyle\int_0^1 e^{-\frac{1}{2}x^2}\,dx \ge 12\left(1 - e^{-\frac{1}{4}}\right)^2$ **A1** **Given Answer** legitimately obtained **5**

(iii) With f(x) = 1, g(x) = $\sqrt{\sin x}$, a = 0, b = $\tfrac{1}{2}\pi$, **M1** Correct choice for f, g (or v.v.)

(*) becomes **M1** Any sensible f, g used in **(*)**

$$\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,dx\right)^2 \le \tfrac{1}{2}\pi\left(\int_0^{\frac{1}{2}\pi} \sin x\,dx\right)$$ **A1**

RHS is $\tfrac{1}{2}\pi\left[-\cos x\right]_0^1 = \tfrac{1}{2}\pi$

(and since LHS is positive) we have $\displaystyle\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,dx \le \sqrt{\frac{\pi}{2}}$

A1 RH half of **Given** inequality obtained from fully correct working **4**

With f(x) = cos x, g(x) = $\sqrt[4]{\sin x}$, $a = 0$, $b = \frac{1}{2}\pi$, **M1** Correct choice for f, g (or v.v.)

(*) gives

$$\left(\int_0^{\frac{1}{2}\pi} \cos x.(\sin x)^{\frac{1}{4}}\,dx\right)^2 \le \left(\int_0^{\frac{1}{2}\pi} \cos^2 x\,dx\right)\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,dx\right)$$ **A1**

$$\text{LHS} = \left(\left[\tfrac{4}{5}(\sin x)^{\frac{5}{4}}\right]_0^{\frac{1}{2}\pi}\right)^2 = \tfrac{16}{25}$$ **M1 A1** By recognition/substitution integration

$$\text{and} \quad \int_0^{\frac{1}{2}\pi} \cos^2 x\,dx = \int_0^{\frac{1}{2}\pi} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)dx$$ **M1**

$$= \left(\left[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\frac{1}{2}\pi}\right)^2 = \tfrac{1}{4}\pi$$ **A1**

Giving the required LH half of the **Given** inequality:

$$\tfrac{16}{25} \le \tfrac{1}{4}\pi\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,dx\right) \quad \text{i.e.} \quad \int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,dx \ge \frac{64}{25\pi}$$

6

Withhold the last A mark if final result is not arrived at

Question 5

(i) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{2a}{2at} = \dfrac{1}{t}$ — **M1** Finding gradt. of tgt. (or by implicit diffn.)

$\Rightarrow$ Grad. nml. at P is $-p$ — **A1**

$\Rightarrow$ Eqn. nml. to C at P is $x - 2ap = -p(x - ap^2)$ — **B1** **FT** any form, e.g. $y = -px + ap(2 + p^2)$

Nml. meets C again when $x = an^2$, $y = 2an$ — **M1** Substd. into nml. eqn.

$\Rightarrow$ $2an = -pan^2 + ap(2 + p^2)$

$\Rightarrow$ $0 = pn^2 + 2n - p(2 + p^2)$

$\Rightarrow$ $0 = (n - p)(pn + [2 + p^2])$ — **M1** Solving attempt

Since $n = p$ at P, it follows that $n = -\dfrac{2 + p^2}{p}$ at N

i.e. $n = -\left(p + \dfrac{2}{p}\right)$ — **A1** **Given Answer** legitimately obtained **6**

(ii) Distance $P(ap^2, 2ap)$ to $N(an^2, 2an)$ is given by

$PN^2 = \left[a(p^2 - n^2)\right]^2 + \left[2a(p - n)\right]^2$ — **M1**

$= a^2(p - n)^2\left\{(p + n)^2 + 4\right\}$

$= a^2\left(2p + \dfrac{2}{p}\right)^2\left\{\left(\dfrac{-2}{p}\right)^2 + 4\right\}$ — **M1** Substituting for n

$= 16a^2\left(\dfrac{p^2 + 1}{p}\right)^2\left\{\dfrac{1 + p^2}{p}\right\} = 16a^2\dfrac{(p^2 + 1)^3}{p^4}$ — **A1** **Given Answer** legitimately obtained **3**

$\dfrac{d(PN^2)}{dp} = 16a^2\dfrac{d(p^2 + 3 + 3p^{-2} + p^{-4})}{dp}$ — **M1** Differentiation directly, or by the Quotient Rule

$= 16a^2(2p - 6p^{-3} - 4p^{-5})$ — **A1** Correct, unsimplified

$= 32a^2\dfrac{p^6 - 3p^2 - 2}{p^5}$

$= \dfrac{32a^2}{p^5}(p^2 + 1)^2\left[p^2 - 2\right]$

Note that $\dfrac{d(PN^2)}{dp} = 16a^2\left\{\dfrac{p^4.3(p^2 + 1)^2.2p - (p^2 + 1)^3.4p^3}{p^8}\right\}$

$= \dfrac{32a^2}{p^8}.p^3(p^2 + 1)^2\left[3p^2 - 2(p^2 + 1)\right]$ by the Quotient Rule

$\dfrac{d(PN^2)}{dp} = 0$ only when $p^2 = 2$ — **A1** **Given Answer** fully shown

Justification that it is a minimum — **E1**

(either by examining the sign of $\dfrac{d(PN^2)}{dp}$

or by explaining that PN^2 cannot be maximised **4**

(iii) Grad. PQ is $\dfrac{2}{p+q}$ **B1**

Grad. NQ is $\dfrac{2}{n+q}$ or $\dfrac{2}{q-p-\frac{2}{p}}$ **B1**

Since $\angle PQN = 90°$ (by "$\angle$ in a semi-circle"; i.e. *Thales Theorem*)

$$\frac{2}{p+q}\times\frac{2}{q-p-\frac{2}{p}}=-1$$ **M1**

$$\Rightarrow\ 4=(p+q)\left(p-q+\frac{2}{p}\right)=p^2-q^2+2+\frac{2q}{p}$$

$$\Rightarrow\ 2=p^2-q^2+\frac{2q}{p}$$ **A1 Given Answer** legitimately obtained **4**

PN minimised when $p^2=2\ \Rightarrow\ q^2=\dfrac{2q}{p}$ **M1** Substituted into given expression

$\Rightarrow\ q=0$ or $q=\dfrac{2}{p}=\pm\sqrt{2}$ **A1**

But $q=\pm\sqrt{2}\ \Rightarrow q=p$ (which is not the case) **E1** Other cases must be ruled out

Special Case: 1/3 for substg. $q = 0$ and verifying that $p^2 = 2$ **3**

Question 6

(i)		
When $n = 1$ $$S_1 = 1 \leq 2\sqrt{1} - 1$$	B1	Clear verification.
Assume that the statement is true when $n = k$: $$S_k \leq 2\sqrt{k} - 1$$	B1	Must be clear that this is assumed.
Then $$S_{k+1} = S_k + \frac{1}{\sqrt{k+1}}$$	M1	Linking S_{k+1} and S_k
$$\leq 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}}$$	M1	Using assumed result
Sufficient to prove: $$2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} - 1$$	M1	
i.e. $2\sqrt{k(k+1)} + 1 \leq 2(k+1)$	A1	Multiplying by $\sqrt{k+1}$ or putting over a common denominator
i.e. $2\sqrt{k(k+1)} \leq 2k + 1$		
i.e. $4k^2 + 4k \leq 4k^2 + 4k + 1$	A1	
Which is clearly true. Therefore by induction the statement is true for all $n \geq 1$.	B1	Clear conclusion showing logic of induction.
	[8]	
(ii)		
Required to prove: $$(4k+1)^2(k+1) > (4k+3)^2 k$$	M2	Squaring given inequality
i.e. $16k^3 + 24k^2 + 9k + 1 > 16k^3 + 24k^2 + 9k$ which is clearly true.	A1	
	[3]	
When $n = 1$: $$S_1 = 1 \geq 2 + \frac{1}{2} - c$$	M1	
So we need $c \geq \frac{3}{2}$	A1	
Prove $c = \frac{3}{2}$ works using induction	M1	
Assume holds when $n = k$: $$S_k \geq 2\sqrt{k} + \frac{1}{2\sqrt{k}} - \frac{3}{2}$$	M1	Allow a general c.
Then $$S_{k+1} = S_k + \frac{1}{\sqrt{k+1}} \geq 2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} - c$$	M1	
Sufficient to prove: $$2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} - c \geq 2\sqrt{k+1} + \frac{1}{2\sqrt{k+1}} - c$$	A1	
i.e. $4k\sqrt{k+1} + \sqrt{k+1} + 2\sqrt{k} \geq 4\sqrt{k}(k+1) + \sqrt{k}$	A1A1	
Which simplifies to the previously proved inequality. No further restrictions on c, so the minimum value is $c = \frac{3}{2}$	B1	
	[9]	

Question 7

(i) For $0 < x < 1$, x is positive and $\ln x$ is negative
so $0 > x \ln x > \ln x$
$\Rightarrow$ $e^0 > e^{x \ln x} > e^{\ln x}$ or $\ln 1 > \ln x^x > \ln x$
$\Rightarrow$ $(1 >)\ f(x) > x$ since ln is a strictly increasing fn. **B1**
Again, since $\ln x < 0$, it follows that
$\ln x < f(x) \ln x < x \ln x$
$\Rightarrow$ $\ln x < \ln\{g(x)\} < \ln\{f(x)\}$ — **M1** Suitably coherent justification
$\Rightarrow$ $x < g(x) < f(x)$ — **A1** **Given Answer** legitimately obtained

For $x > 1$, $\ln x > 0$ and so $x < f(x) < g(x)$ — **B1** No justification required — **4**

(ii) $\ln\{f(x)\} = x \ln x$ — **M1** Taking logs and attempting implicit diffn.
Alt. Writing $y = e^{x \ln x}$ and diffg.

$\frac{1}{f(x)}.f'(x) = x.\frac{1}{x} + 1.\ln x$ i.e. $f'(x) = (1 + \ln x)f(x)$ — **A1**

$f'(x) = 0$ when $1 + \ln x = 0$, $\ln x = -1$, $x = e^{-1}$ — **A1** — **3**

(iii) $\lim_{x \to 0}(f(x)) = \lim_{x \to 0}\left(e^{x \ln x}\right) = \lim_{x \to 0}\left(e^0\right) = 1$ — **B1** Suitably justified

$\lim_{x \to 0}(g(x)) = \lim_{x \to 0}\left(x^{f(x)}\right) = \lim_{x \to 0}\left(x^1\right) = 0$ — **B1** May just be stated

Alt. $\lim_{x \to 0}(g(x)) = \lim_{x \to 0}\left(e^{f(x)\ln x}\right) = \lim_{x \to 0}\left(e^{\ln x}\right) = \lim_{x \to 0}(x) = 0$ — **2**

(iv) For $y = \frac{1}{x} + \ln x$ $(x > 0)$,

$\frac{dy}{dx} = -\frac{1}{x^2} + \frac{1}{x}$ or $\frac{x-1}{x^2} = 0$ … — **M1** Diffg. and equating to zero

… when $x = 1$ — **A1** From correct derivative

For $x = 1-$, $\frac{dy}{dx} < 0$ and for $x = 1+$, $\frac{dy}{dx} > 0$ — **M1** Method for deciding

(1, 1) is a MINIMUM of $y = \frac{1}{x} + \ln x$ — **A1**

(Since there are no other TPs or discontinuities)
$y \geq 1$ for all $x > 0$ — Conclusion must be made for all 4 marks — **4**

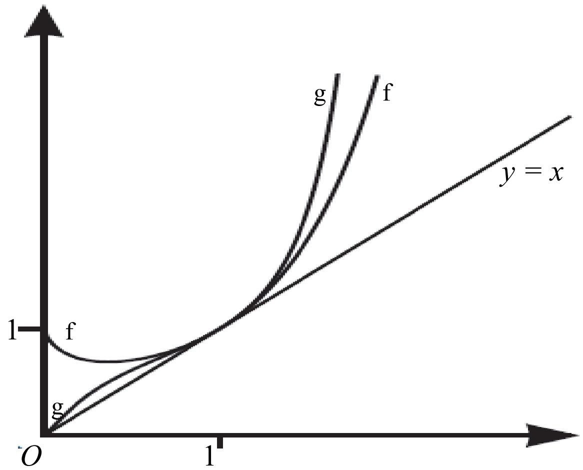
$\ln(g(x)) = f(x) \ln x$ — **M1** Taking logs and attempting implicit diffn.

$\frac{1}{g(x)}.g'(x) = f(x).\frac{1}{x} + \ln x\{f(x)(1 + \ln x)\}$ — **A1** using $f'(x)$ from **(ii)**

$\Rightarrow$ $g'(x) = f(x).g(x)\left\{\frac{1}{x} + \ln x + (\ln x)^2\right\}$

$\geq f(x).g(x)\{1 + (\ln x)^2\}$ — **M1** using previous result of **(iv)**

> 0 since f, g > 0 from **(i)**
and $1 + (\ln x)^2 \geq 1 > 0$ — **A1** **Given Answer** fully justified — **4**



B1 One of f, g correct …
B1 Both correct …
… relative to $y = x$

B1 All three passing thro' (1, 1)

3

Question 8

Line thro' A perpr. to BC is $\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{u}$	**B1**
Line thro' B perpr. to CA is $\mathbf{r} = \mathbf{b} + \mu\,\mathbf{v}$	**B1**
Lines meet when $(\mathbf{r} = \mathbf{p} =)\ \mathbf{a} + \lambda\,\mathbf{u} = \mathbf{b} + \mu\,\mathbf{v}$	**M1** Equated
$\Rightarrow\ \mathbf{v} = \dfrac{1}{\mu}\left(\mathbf{a} - \mathbf{b} + \lambda\,\mathbf{u}\right)$	**A1**
Since $\mathbf{v}$ is perpr. to CA, $\left(\mathbf{a} - \mathbf{b} + \lambda\,\mathbf{u}\right)\bullet(\mathbf{a} - \mathbf{c}) = 0$ (boxed)	**M1**
$\Rightarrow\ (\mathbf{a}-\mathbf{b})\bullet(\mathbf{a}-\mathbf{c}) + \lambda\,\mathbf{u}\bullet(\mathbf{a}-\mathbf{c}) = 0$	**A1** Correctly multiplied out
$\Rightarrow\ \lambda = \dfrac{(\mathbf{b}-\mathbf{a})\bullet(\mathbf{a}-\mathbf{c})}{\mathbf{u}\bullet(\mathbf{a}-\mathbf{c})}$	**M1** Re-arranging for λ
	A1 Correct (any sensible form)
$\Rightarrow\ \mathbf{p} = \mathbf{a} + \left(\dfrac{(\mathbf{b}-\mathbf{a})\bullet(\mathbf{a}-\mathbf{c})}{\mathbf{u}\bullet(\mathbf{a}-\mathbf{c})}\right)\mathbf{u}$	**A1** **FT** their λ (if only **a**, **b**, **c**, **u** involved)

9

$\overrightarrow{CP} = \mathbf{p} - \mathbf{c} = \mathbf{a} - \mathbf{c} + \lambda\mathbf{u}$	**B1** **FT** their λ
Attempt at $\overrightarrow{CP}\bullet\overrightarrow{AB}$	**M1**
$= (\mathbf{a}-\mathbf{c}+\lambda\mathbf{u})\bullet(\mathbf{b}-\mathbf{a})$	**A1** Correct to here
$= (\mathbf{a}-\mathbf{c})\bullet(\mathbf{b}-\mathbf{a}) + \lambda\mathbf{u}\bullet(\mathbf{b}-\mathbf{a})$	
Now $\mathbf{u}\bullet(\mathbf{b}-\mathbf{c}) = 0$ since $\mathbf{u}$ perpr. to BC	**M1**
$\Rightarrow\ \mathbf{u}\bullet\mathbf{b} = \mathbf{u}\bullet\mathbf{c}$	**A1**
so that $\overrightarrow{CP}\bullet\overrightarrow{AB} = (\mathbf{a}-\mathbf{c})\bullet(\mathbf{b}-\mathbf{a}) + \lambda\mathbf{u}\bullet(\mathbf{c}-\mathbf{a})$	**M1** Substituted in
$= (\mathbf{a}-\mathbf{c})\bullet(\mathbf{a}-\mathbf{b}+\lambda\mathbf{u})$	**M1 A1** Factorisation attempt; correct
$= 0$ from boxed line above	**A1 E1** Statement; justified
$\Rightarrow\ CP$ is perpr. to AB	**E1** For final, justified statement

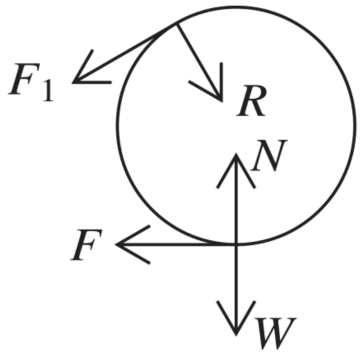
11

Notice that the "value" of   is never actually required

Any candidate who states the result is true because P is the *orthocentre* of ΔABC may be awarded **B2** for actually knowing something about triangle-geometry, but only in addition to any of the first 3 marks earned in the above solution: i.e. a maximum of 5/11 for the second part of the question.

Question 9

(i)



$O \circlearrowleft$ $\quad F\,.\,r = F_1\,.\,r$ $\Rightarrow F = F_1$	**B1**	For correct moment equation.
Res. $\leftrightarrow$ $F + F_1\cos\theta = R\sin\theta$	**B1**	For resolving horizontally for one cylinder.
Together give $R\sin\theta = F(1 + \cos\theta)$	**AG**	
Since $F_1 \le \mu R$, with $\mu = \frac{1}{2}$,	**M1**	Use of the Friction law
it follows that $\frac{F}{R} \le \frac{1}{2} \Rightarrow \frac{\sin\theta}{1+\cos\theta} \le \frac{1}{2}$	**A1**	Combining with previous answer
i.e. $2\sin\theta \le 1 + \cos\theta$	**AG** **Subtotal: 4**	
(ii) Res. $\uparrow$ for RH cylinder $W = N - R\cos\theta - F_1\sin\theta$	**B1**	F_1 might correctly be replaced with F.
Res. $\uparrow$ for plank $kW = 2R\cos\theta + 2F\sin\theta$	**B1**	
Eliminating W: $k(N - R\cos\theta - F\sin\theta) = 2R\cos\theta + 2F\sin\theta$	**M1**	For eliminating W
$N = R\cos\theta\left(\frac{2}{k}+1\right) + F\sin\theta\left(\frac{2}{k}+1\right)$	**A1**	For correct rearrangement for N
$N = \left(\frac{2}{k}+1\right)\left(\frac{1+\cos\theta}{\sin\theta}.\cos\theta + \sin\theta\right)F$	**M1**	For use of $R = \left(\frac{1+\cos\theta}{\sin\theta}\right)F$
$N = \left(\frac{2}{k}+1\right)\left(\frac{\cos\theta + \cos^2\theta + \sin^2\theta}{\sin\theta}\right)F$	**B1**	Obtaining $\cos^2\theta + \sin^2\theta$
$= \left(\frac{2}{k}+1\right)\left(\frac{\cos\theta + 1}{\sin\theta}\right)F$	**AG**	
For no slipping at the ground, $F \le \mu N$	**M1**	Using Friction equation
$\Rightarrow F \le \frac{1}{2}\left(\frac{2}{k}+1\right)\left(\frac{\cos\theta+1}{\sin\theta}\right)F$	**A1**	Using previous part
ie. $2k\sin\theta \le (k+2)(1+\cos\theta)$	**A1**	Rearranging into a "useful" form.

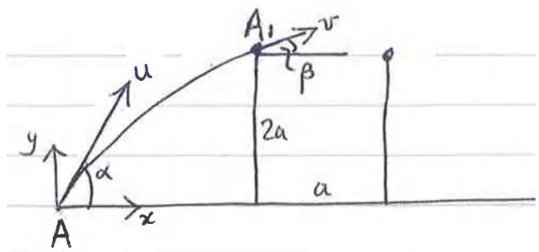
However, we already have that $2k\sin\theta \le k(1+\cos\theta) \le (k+2)(1+\cos\theta)$ so there are no extra restrictions on θ.	**E1** **Subtotal: 10**	Properly justified
(iii) $4\sin^2\theta \le 1 + 2\cos\theta + cos^2\theta$ $4(1-\cos^2\theta) \le 1 + 2\cos\theta + \cos^2\theta$ $0 \le 5\cos^2\theta + 2\cos\theta - 3$ $0 \le (5\cos\theta - 3)(\cos\theta + 1)$ Since $\cos\theta \ge 0$ we have $\cos\theta \ge \frac{3}{5}$ For appropriate angles $\cos\theta$ is decreasing and $\sin\theta$ is increasing. Therefore $\sin\theta \le \frac{4}{5}$ $\sin\theta = \frac{r-a}{r}$ So $5r - 5a \le 4r$ $r \le 5a$	**M1** **M1** **A1** **E1** **AG** **B1** **M1** **AG** **Subtotal: 6**	Squaring up an appropriate trig inequality Creating and simplifying quadratic inequality in one trig ratio A graphical argument is perfectly acceptable here. N.b It is possible that inequalities like $2s - 1 \le c$ are squared. If this is done without justifying that both sides are positive then withhold this final **E1**. Combining with previous result

Question 10

$ma = F - (Av^2 + R)$ $\text{WD} = \int_0^d F\,\mathrm{d}x$ $= \int_0^d \left(ma + Av^2 + R\right)\mathrm{d}x$ Since $a = v\dfrac{\mathrm{d}v}{\mathrm{d}x}$ $\text{WD} = \int_{x=0}^{x=d} \left(ma + Av^2 + R\right)\dfrac{\mathrm{d}x}{\mathrm{d}v}\,\mathrm{d}v$ $= \int_{x=0}^{x=d} \left(ma + Av^2 + R\right)\dfrac{v}{a}\,\mathrm{d}v$ Using $v^2 = u^2 + 2as$ with $v = w$, $u = 0$, $s = d \Rightarrow$ $w = \sqrt{2ad}$ Therefore: $\text{WD} = \int_{v=0}^{v=w} \dfrac{\left(ma + Av^2 + R\right)v}{a}\,\mathrm{d}v$	**B1** **M1** **AG** **B1** **M1** **B1** **AG** **[5]**	Clear use of N2L Attempting to change variable of integration. Justifying limits. Ignore absence of ±
(i) $\text{WD} = \left[\left(m + \dfrac{R}{a}\right)\dfrac{v^2}{2} + \dfrac{Av^4}{4a}\right]_0^{\sqrt{2ad}}$ $= \left(m + \dfrac{R}{a}\right)ad + Aad^2$ For second half-journey, $\text{WD} = \int_w^0 \dfrac{\left(-ma + Av^2 + R\right)v}{-a}\,\mathrm{d}v$ $= -mad + Rd + Aad^2$ Summing gives $2dR + 2Aad^2$ $R > ma \Rightarrow F = Av^2 + R - ma > 0$ always	**M1** **A1** **B1B1** **A1** **AG** **E1** **[6]**	Performing integration Correct answer in terms of d. B1 for correct limits B1 for correct integrand N.b. integrals may be combined to get to the same result.

(ii) If $R < ma$ then F is zero when $Av^2 = ma - R$ i.e. when $v = V = \sqrt{\dfrac{ma-R}{A}}$	**B1**	Finding an expression for the critical speed.
For F to fall to zero during motion, $V < w$ i.e. when $\dfrac{ma-R}{A} < 2ad$ i.e. $R > ma - 2Aad$	**E1** **E1**	
In this case, WD = $mad + Rd + Aad^2$, as before, for the first half-journey	**B1**	
For the second half WD = $\displaystyle\int_w^V \frac{\left(-ma + Av^2 + R\right)v}{-a}\, \mathrm{d}v$	**M2**	
$\left[(ma-R)\dfrac{v^2}{2a} - \dfrac{Av^4}{4a}\right]_w^V$	**A1**	
$= \dfrac{1}{2a}(ma-R)\left(\dfrac{ma-R}{A}\right) - \dfrac{A}{4a}\left(\dfrac{ma-R}{A}\right)^2 -$ $\dfrac{1}{2a}(ma-R)(2ad) + \dfrac{A}{4a}\left(4a^2d^2\right)$ $= \dfrac{1}{2Aa}(ma-R)^2 - \dfrac{1}{4Aa}(ma-R)^2 - (ma-R)d +$ Aad^2	**M1**	Substituting expressions for V and w.
$= \dfrac{1}{4Aa}(ma-R)^2 - mad + Rd + Aad^2$	**A1 CAO**	Without wrong working
So total WD = $\dfrac{1}{4Aa}(ma-R)^2 + 2Rd + 2Aad^2$	**AG** **[9]**	

Question 11



(i)		
At A, $KE = \frac{1}{2}mu^2 = \frac{5}{2}mag, PE = 0$	**B1**	
At $A_1, K = \frac{1}{2}mv^2, PE = 2mag$	**B1**	
Conservation of energy: $$\frac{5}{2}mag = \frac{1}{2}mv^2 + 2mag$$	**M1**	
$v^2 = ga$		
$v = \sqrt{ga}$	**A1**	
	[4]	
If angle at A_1 is β and it just passes the second wall then we have:		
$0 = v\sin\theta\, t - \frac{1}{2}gt^2$	**M1**	Using $s = ut + \frac{1}{2}at^2$
So $t = \frac{2v}{g}\sin\beta$	**A1**	Solving for t at second wall.
Also, $a = v\cos\beta\, t$	**M1**	Considering horizontal distance
$= \frac{2v^2\sin\beta\cos\beta}{g}$		N.b. Some candidates may just quote this (or equivalent). Give full credit.
$= 2a\sin\beta\cos\beta$	**A1**	Combining previous results.
So $\sin(2\beta) = 1$	**A1**	
Therefore $\beta = 45°$	**AG**	Condone absence of domain considerations.
	[5]	
x velocity is constant so		
$u\cos\alpha = vcos\,\beta$	**M1**	Comparing x velocities
$\sqrt{5ag}\cos\alpha = \sqrt{ag}\frac{1}{\sqrt{2}}$ $\cos\alpha = \frac{1}{\sqrt{10}}$	**A1**	
$\sin\alpha = \frac{3}{\sqrt{10}}, \tan\alpha = 3$	**A1**	Converting to a more useful ratio.

Method 1: $$2a = \sqrt{5ag}\frac{3}{\sqrt{10}}t - \frac{1}{2}gt^2$$ $$= \frac{3\sqrt{ag}}{\sqrt{2}}t - \frac{1}{2}gt^2$$ So $$t^2 - \frac{3\sqrt{2a}}{\sqrt{g}}t + \frac{4a}{g} = 0$$	**M1**	Using $s = ut + \frac{1}{2}at^2$
$$\left(t - \sqrt{\frac{2a}{g}}\right)\left(t - 2\sqrt{\frac{2a}{g}}\right) = 0$$		
First time over the wall means that $t = \sqrt{\frac{2a}{g}}$	**A1**	
So $d = ucos\,\theta\, t = \sqrt{5ag} \times \frac{1}{\sqrt{10}} \times \sqrt{\frac{2a}{g}} = a$	**A1**	
Method 2: $$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2}$$	**M1**	Using trajectory equation
$$2a = 3x - \frac{x^2}{a}$$	**A1**	Combining with previous results
$$(x-a)(x-2a) = 0$$		
$$x = a$$	**A1**	
	[6]	
If the speed at h above first wall is v then by conserving energy, $$\frac{1}{2}5ag = \frac{1}{2}v^2 + (2a+h)g$$	**M1**	
$$v^2 = ag - 2gh$$	**B1**	
Using trajectory equation with origin at top of first wall and angle β as particle moves over first wall: $$y = h + x\ \tan\beta - \frac{gx^2(1+\tan^2\beta)}{2v^2}$$ When $x = a$ we need $y = 0$: $$0 = h + a\ \tan\beta - \frac{ga^2(1+\tan^2\beta)}{2v^2}$$	**M1**	Use of trajectory equation (might be several kinematics equations effectively leading to the same thing)
Treating this as a quadratic in $\tan\beta$: $$-\frac{ga^2}{2v^2}\tan^2\beta + a\tan\beta + h - \frac{ga^2}{2v^2} = 0$$ $$-ga^2\tan^2\beta + 2av^2\tan\beta + 2hv^2 - ga^2 = 0$$ The discriminant is: $$4a^2v^4 + 4ga^2(2hv^2 - ga^2)$$	**M1**	Considering the quadratic (or equivalently differentiating to find the max)
$$= 4a^2(g^2(a^2 - 4ah + 4h^2) + 2g^2h(a-2h) - g^2a^2))$$ $$= 4a^2g^2(a^2 - 4ah + 4h^2 + 2ah - 4h^2 - a^2)$$ $$= -8a^3g^2h$$ $$< 0$$ Therefore no solution.	**A1**	Obtaining a clearly negative discriminant – this might take many alternative forms.
	[5]	

Question 12

(i) $$P(X+Y=n)=\sum_{r=0}^{n} P(X=r)(P(Y=n-r)$$	**B2**	
$$=\sum_{r=0}^{n}\frac{e^{-\lambda}\lambda^{r}}{r!}\times\frac{e^{-\mu}\mu^{n-r}}{(n-r)!}$$	**B1**	
$$=\frac{e^{-\lambda}e^{-\mu}}{n!}\sum_{r=0}^{n}\frac{n!}{r!\,(n-r)!}\lambda^{r}\mu^{n-r}$$	**M1**	Attempting to manipulate factorials towards a binomial coefficient
$$=\frac{e^{-\lambda}e^{-\mu}}{n!}\sum_{r=0}^{n}\binom{n}{r}\lambda^{r}\mu^{n-r}$$	**B1**	Identifying correct binomial coefficient
$$=\frac{e^{-(\lambda+\mu)}}{n!}(\lambda+\mu)^{n}$$	**B1**	
Which is the the formula for $Po(\lambda+\mu)$	**E1**	Recognising result. Must state parameters
	[7]	
(ii)		
$$P(X=r\mid X+Y=k)=\frac{P(X=r)\times P(Y=k-r)}{P(X+Y=k)}$$	**M2**	(may be implied by following line)
$$=\frac{\frac{e^{-\lambda}\lambda^{r}}{r!}\times\frac{e^{-\mu}\mu^{k-r}}{(k-r)!}}{\frac{e^{-(\lambda+\mu)}}{k!}(\lambda+\mu)^{k}}$$	**A1**	
$$=\frac{k!}{r!\,(k-r)!}\left(\frac{\lambda}{\lambda+\mu}\right)^{r}\left(\frac{\mu}{\lambda+\mu}\right)^{k-r}$$	**A1**	
Which is a $B\left(k,\frac{\lambda}{\lambda+\mu}\right)$ distribution.	**E1**	Parameters must be stated.
	[5]	
(iii) This corresponds to r=1, k=1 from (ii)	**M2**	Can be implied by correct answer.
So probability is $\frac{\lambda}{\lambda+\mu}$.	**A1**	
(iv)	**[3]**	
Expected waiting time given that Adam is first is waiting time for first fish plus waiting time for Eve $\left(=\frac{1}{\lambda+\mu}+\frac{1}{\mu}\right)$	**B2**	Also accept waiting time given Eve is first. Must be clearly identified.
Expected waiting time is: E(Waiting time|Adam first)P(Adam first)+E(Waiting time|Eve first)P(Eve first)	**M2**	
$=\left(\frac{1}{\lambda+\mu}+\frac{1}{\mu}\right)\times\frac{\lambda}{\lambda+\mu}+\left(\frac{1}{\lambda+\mu}+\frac{1}{\lambda}\right)\times\frac{\mu}{\lambda+\mu}$	**A1**	
$$=\frac{1}{\lambda}+\frac{1}{\mu}-\frac{1}{\lambda+\mu}$$		No need for this algebraic simplification.
	[5]	

Question 13

(i)		
$P(\text{correct key on } k^{\text{th}}\text{attempt}) = \frac{1}{n}\left(1-\frac{1}{n}\right)^{k-1}$	**M1A1**	M1 for any attempt relating to the geometric distribution – e.g. missing first factor or power slightly wrong.
$= pq^{k-1}$ Where $p = \frac{1}{n}, q = 1-\frac{1}{n}$		Although not strictly necessary, you may see this substitution frequently
Expected number of attempts is given by $p + 2pq + 3pq^2 \dots$	**M1**	May be written in sigma notation
$= p(1 + 2q + 3q^2 \dots)$		
$= p(1-q)^{-2}$	**M1**	Linking to binomial expansion
$= \frac{p}{p^2} = \frac{1}{p}$		
$= n$	**A1**	
	[5]	
(ii)		
$P(\text{correct key on } k^{\text{th}}\text{attempt}) = \frac{1}{n}$ for $k = 1 \dots n$	**B1**	
Expected number of attempts is given by $\frac{1}{n} + \frac{2}{n} + \frac{3}{n} \dots + \frac{n}{n}$	**M1**	
$= \frac{n+1}{2}$	**M1A1**	M1 for clearly recognising sum of integers / arithmetic series.
	[4]	
(iii)		
$P(\text{correct key on } k^{\text{th}}\text{attempt})$ $= \frac{n-1}{n} \times \frac{n}{n+1} \times \frac{n+1}{n+2} \dots \times \frac{1}{n+k-1}$	**M1** **A1**	M1 for an attempt at this, possibly by pattern spotting the first few cases. Condone absence of checking $k = 1$ case explicitly.
$= \frac{n-1}{(n+k-2)(n+k-1)}$	**M1** **AG**	M1 for attempting telescoping (may be written as an induction)
$= (n-1)\left(\frac{-1}{n+k-1} + \frac{1}{n+k-2}\right)$	**M2** **A1**	Attempting partial fractions (This may be seen later)
	[6]	
Expected number of attempts is given by $(n-1)\sum_{1}^{\infty}\left(\frac{k}{n+k-2} - \frac{k}{n+k-1}\right)$	**M1**	
$= (n-1)\left[\left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{2}{n} - \frac{2}{n+1}\right) + \left(\frac{3}{n+1} - \frac{3}{n+2}\right) \dots\right]$		
$= (n-1)\left[\frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} \dots\right]$	**M1A1**	M1 for attempting telescoping
$= (n-1)\left(\sum_{r=1}^{\infty}\frac{1}{r} - \sum_{r=1}^{n-2}\frac{1}{r}\right)$	**B1**	
In the brackets there is an infinite sum minus a finite sum, so the result is infinite.	**E1**	
	[5]	

Question 1

(i) $\frac{1}{{}^{n+r-1}C_r} - \frac{1}{{}^{n+r}C_r} = \frac{(n-1)!\,r!}{(n+r-1)!} - \frac{n!r!}{(n+r)!}$ **M1**

$= \frac{(n-1)!\,r!\,[(n+r)-n]}{(n+r)!}$

$= \frac{(n-1)!\,r!\,r}{(n+r)!}$ **M1**

$\therefore \frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}C_r} - \frac{1}{{}^{n+r}C_r}\right) = \frac{r+1}{r}\frac{(n-1)!\;r!\;r}{(n+r)!}$

$= \frac{(n-1)!\,(r+1)!}{(n+r)!} = \frac{1}{{}^{n+r}C_{r+1}}$ **A1*** **(3)**

$\sum_{n=1}^{\infty} \frac{1}{{}^{n+r}C_{r+1}} = \sum_{n=1}^{\infty} \frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}C_r} - \frac{1}{{}^{n+r}C_r}\right)$ **M1**

$= \frac{r+1}{r}\left(\frac{1}{{}^{r}C_r} - \frac{1}{{}^{r+1}C_r} + \frac{1}{{}^{r+1}C_r} - \frac{1}{{}^{r+2}C_r} + \frac{1}{{}^{r+2}C_r} - \frac{1}{{}^{r+3}C_r} + \cdots\right)$ **M1**

$= \frac{r+1}{r}\frac{1}{{}^{r}C_r}$ because ${}^{n+r}C_r \to \infty$ as $n \to \infty$ **E1**

$= \frac{r+1}{r}$ **A1** **(4)**

$\sum_{n=2}^{\infty} \frac{1}{{}^{n+2}C_{2+1}} = \frac{2+1}{2} - \frac{1}{{}^{1+2}C_{2+1}} = \frac{3}{2} - 1 = \frac{1}{2}$ **M1 M1 (2)**

(ii) ${}^{n+1}C_3 = \frac{(n+1)!}{(n-2)!3!} = \frac{(n+1)n(n-1)}{3!} = \frac{n^3-n}{3!} < \frac{n^3}{3!}$ **M1**

So $\frac{3!}{n^3} < \frac{1}{{}^{n+1}C_3}$ **A1*** **(2)**

$\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} - \frac{5!}{n^3} = \frac{120}{n(n^2-1)} - \frac{120}{n(n^2-1)(n^2-4)} - \frac{120}{n^3}$

$= \frac{120}{n^3(n^2-1)(n^2-4)}\left(n^2(n^2-4) - n^2 - (n^2-1)(n^2-4)\right)$ **M1**

$= \frac{-480}{n^3(n^2-1)(n^2-4)} < 0$

as $n \geq 3$ and so denominator is positive. **E1** **(2)**

Hence, $\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} < \frac{5!}{n^3}$

Alternatively,

$\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} = \frac{5!}{n(n^2-1)} - \frac{5!}{n(n^2-1)(n^2-4)} = \frac{5!}{n(n^2-1)(n^2-4)} \times \left((n^2-4)-1\right)$

$= \frac{5!}{n^3} \times \frac{n^4-5n^2}{n^4-5n^2+4} < \frac{5!}{n^3}$

as $n \geq 3$ and so $n^2 > 5$

$$\sum_{n=3}^{\infty}\frac{3!}{n^3} < \sum_{n=3}^{\infty}\frac{1}{{}^{n+1}C_3} = \sum_{n=2}^{\infty}\frac{1}{{}^{n+2}C_3} = \frac{1}{2}$$

M1

So $\sum_{n=1}^{\infty}\frac{3!}{n^3} < \frac{3!}{1}+\frac{3!}{8}+\frac{1}{2}=\frac{29}{4}$

M1

And therefore $\sum_{n=1}^{\infty}\frac{1}{n^3} < \frac{29}{24}=\frac{116}{96}$

A1* (3)

$$\sum_{n=3}^{\infty}\frac{5!}{n^3} > \sum_{n=3}^{\infty}\left(\frac{20}{{}^{n+1}C_3}-\frac{1}{{}^{n+2}C_5}\right) = 20\times\frac{1}{2}-\left(\sum_{n=1}^{\infty}\frac{1}{{}^{n+4}C_5}\right) = 10-\frac{5}{4}$$

M1 **M1**

Therefore

$$\sum_{n=3}^{\infty}\frac{5!}{n^3} > \frac{35}{4}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^3} > \frac{35}{4\times 5!}+\frac{1}{1}+\frac{1}{8}=\frac{7}{96}+\frac{96}{96}+\frac{12}{96}=\frac{115}{96}$$

M1 **A1* (4)**

Question 2

(i) $z' - a = e^{i\theta}(z - a)$ **M1**

Thus $z' = a + e^{i\theta}z - e^{i\theta}a = e^{i\theta}z + a(1 - e^{i\theta})$ **A1* (2)**

(ii) $z'' = e^{i\varphi}z' + b(1 - e^{i\varphi})$

$= e^{i\varphi}\left(e^{i\theta}z + a(1 - e^{i\theta})\right) + b(1 - e^{i\varphi})$ **M1 A1**

So $z'' = e^{i(\varphi+\theta)}z + \left(ae^{i\varphi} - ae^{i(\varphi+\theta)} + b - be^{i\varphi}\right)$

This is a rotation about c if

$c\left(1 - e^{i(\varphi+\theta)}\right) = ae^{i\varphi} - ae^{i(\varphi+\theta)} + b - be^{i\varphi}$ **M1**

If $\varphi + \theta = 2n\pi$, $\left(1 - e^{i(\varphi+\theta)}\right) = 0$, so c cannot be found. **E1**

Otherwise, multiplying by $-e^{\frac{-i(\varphi+\theta)}{2}}$,

$c\left(e^{\frac{i(\varphi+\theta)}{2}} - e^{\frac{-i(\varphi+\theta)}{2}}\right) = a\left(e^{\frac{i(\varphi+\theta)}{2}} - e^{\frac{i(\varphi-\theta)}{2}}\right) + b\left(e^{\frac{i(\varphi-\theta)}{2}} - e^{\frac{-i(\varphi+\theta)}{2}}\right)$

$2ci\sin\frac{1}{2}(\varphi + \theta) = 2aie^{i\varphi/2}\sin\frac{1}{2}\theta + 2bie^{-i\theta/2}\sin\frac{1}{2}\varphi$ **M1**

$c\sin\frac{1}{2}(\varphi + \theta) = ae^{i\varphi/2}\sin\frac{1}{2}\theta + be^{-i\theta/2}\sin\frac{1}{2}\varphi$ **A1* (6)**

If $\varphi + \theta = 2n\pi$, $z'' = z + \left(ae^{i\varphi} - a + b - be^{i\varphi}\right)$ **M1**

So $z'' = z + (b - a)\left(1 - e^{i\varphi}\right)$ **A1**

This is a translation by $(b - a)\left(1 - e^{i\varphi}\right)$ **A1 (3)**

(iii) If RS = SR, and if $\varphi + \theta = 2n\pi$, then

$$(b - a)\left(1 - e^{i\varphi}\right) = (a - b)\left(1 - e^{i\theta}\right)$$

M1

$$(a - b)\left(e^{i\theta} + e^{i\varphi} - 2\right) = 0$$

So $a = b$, or if $a \neq b$, $e^{i\theta} + e^{i(2n\pi-\theta)} - 2 = 0$

A1

$2\cos\theta - 2 = 0$ **M1**

Thus $\theta = 2n\pi$ **A1 (4)**

If $\varphi + \theta \neq 2n\pi$

$ae^{i\varphi/2}\sin\frac{1}{2}\theta + be^{-i\theta/2}\sin\frac{1}{2}\varphi = be^{i\theta/2}\sin\frac{1}{2}\varphi + ae^{-i\varphi/2}\sin\frac{1}{2}\theta$ **M1**

$2i(a - b)\sin\frac{1}{2}\varphi\sin\frac{1}{2}\theta = 0$ **A1**

So $a = b$, $\theta = 2n\pi$, or $\varphi = 2n\pi$

A1 **A1** **A1 (5)**

Question 3

$\alpha\beta + \gamma\delta + \alpha\gamma + \beta\delta + \alpha\delta + \beta\gamma = -A$ **M1**

$A = -q$ **A1 (2)**

(i) $y^3 - 3y^2 - 40y + 84 = 0$ **M1 A1**
$(y-2)(y^2 - y - 42) = 0$ **M1**

$(y-2)(y-7)(y+6) = 0$ **M1 A1**

So $\alpha\beta + \gamma\delta = 7$ **A1 (6)**

(ii) $(\alpha+\beta)(\gamma+\delta) = \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta = 3 - \alpha\beta - \gamma\delta = -4$

M1 **M1** **A1 (3)**

$(\alpha+\beta) + (\gamma+\delta) = 0$ **M1**

Thus $(\alpha+\beta)$ is a root of $t^2 - 4 = 0$ **M1**

So $\alpha + \beta = \pm 2$ **A1**

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = 6$$

$$\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = 6$$

$2(\alpha\beta - \gamma\delta) = \pm 6$ **M1**

$\alpha\beta - \gamma\delta = 3$ as $\alpha\beta > \gamma\delta$ (and so $\alpha + \beta = -2$)

So $\alpha\beta = 5$ **A1 (5)**

Alternatively, $\alpha\beta\gamma\delta = 10$, **M1 A1** so $\alpha\beta$ and $\gamma\delta$ are the roots of

$t^2 - 7t + 10 = 0$ **M1 A1** and as $\alpha\beta > \gamma\delta$, $\alpha\beta = 5$ (and $\gamma\delta = 2$). **A1 (5)**

(iii) Thus α and β are the roots of $t^2 + 2t + 5 = 0$ and γ and δ are the roots of
$t^2 - 2t + 2 = 0$ **M1 A1**

So $x = 1 \pm i$, $-1 \pm 2i$ **A1 A1 (4)**

Question 4

(i) $e^{x\ln a} = a^x$ (formula book)

So if $\log_a f(x) = z$

$f(x) = a^z = e^{z\ln a}$ **E1**

and so $\ln f(x) = z\ln a = \ln a\ \log_a f(x)$ **B1**

Therefore,

$$e^{\frac{1}{y}\int_0^y \ln f(x)dx} = e^{\frac{1}{y}\int_0^y \ln a\ \log_a f(x)\,dx} = e^{\frac{1}{y}\ln a\int_0^y \log_a f(x)\,dx}$$

M1 **M1**

Thus, $F(y) = a^{\frac{1}{y}\int_0^y \log_a f(x)\,dx}$ **A1* (5)**

(ii) $H(y) = e^{\frac{1}{y}\int_0^y \ln h(x)dx} = e^{\frac{1}{y}\int_0^y \ln(f(x)g(x))dx}$ **M1**

$$= e^{\frac{1}{y}\int_0^y \ln f(x)+\ln g(x)dx}$$

$= e^{\frac{1}{y}\left(\int_0^y \ln f(x)dx+\int_0^y \ln g(x)dx\right)}$ **M1**

$= e^{\frac{1}{y}\int_0^y \ln f(x)dx}\ e^{\frac{1}{y}\int_0^y \ln g(x)dx} = F(y)G(y)$

M1 **A1* (4)**

(iii) Let $f(x) = b^x$,

Then $F(y) = e^{\frac{1}{y}\int_0^y \ln b^x dx} = e^{\frac{1}{y}\int_0^y x\ln b\ dx} = e^{\frac{1}{y}\ln b\ \int_0^y x dx}$

M1 **M1**

$= e^{\frac{1}{y}\ln b\ \left[\frac{1}{2}x^2\right]_0^y} = e^{\frac{1}{y}\ln b\ \frac{1}{2}y^2} = e^{\frac{1}{2}y\ln b} = b^{\frac{1}{2}y} = \sqrt{b^y}$

A1 **M1** **A1* (5)**

(iv) $e^{\frac{1}{y}\int_0^y \ln f(x)dx} = \sqrt{f(y)}$

$\frac{1}{y}\int_0^y \ln f(x)dx = \ln\sqrt{f(y)} = \frac{1}{2}\ln f(y)$

$\int_0^y \ln f(x)dx = \frac{y}{2}\ln f(y)$ **M1**

$\ln f(y) = \frac{1}{2}\ln f(y) + \frac{yf'(y)}{2f(y)}$ **M1**

$\frac{yf'(y)}{f(y)} = \ln f(y)$ so $\frac{f'(y)}{f(y)\ln f(y)} = \frac{1}{y}$ **M1**

Integrating $\ln\ln f(y) = \ln y + c = \ln y + \ln k = \ln ky$ **M1 A1**

$\ln f(y) = ky$

$f(y) = e^{ky} = e^{y\ln b} = b^y$

$f(x) = b^x$ **A1* (6)**

Question 5

$y = r\sin\theta$

$\frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta = f\cos\theta + f'\sin\theta$ **M1**

$x = r\cos\theta$

$\frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta = -f\sin\theta + f'\cos\theta$ **M1**

$\frac{dy}{dx} = \frac{f\cos\theta + f'\sin\theta}{-f\sin\theta + f'\cos\theta} = \frac{f + f'\tan\theta}{-f\tan\theta + f'}$ **M1 A1 (4)**

$\frac{f + f'\tan\theta}{-f\tan\theta + f'} \times \frac{g + g'\tan\theta}{-g\tan\theta + g'} = -1$ **M1**

$fg + f'g\tan\theta + fg'\tan\theta + f'g'\tan^2\theta = -fg\tan^2\theta + f'g\tan\theta + fg'\tan\theta - f'g'$

$(fg + f'g')\sec^2\theta = 0$ **M1**

$fg + f'g' = 0$ **A1* (3)**

$g(\theta) = a(1 + \sin\theta)$

$g'(\theta) = a\cos\theta$

So $f'a\cos\theta + fa(1 + \sin\theta) = 0$ **M1**

$\frac{f'}{f} = -\frac{(1+\sin\theta)}{\cos\theta} = -\sec\theta - \tan\theta$ **A1**

$\ln f = -\ln(\sec\theta + \tan\theta) + \ln\cos\theta + c = \ln\left(\frac{k\cos\theta}{\sec\theta + \tan\theta}\right) = \ln\left(\frac{k\cos^2\theta}{1+\sin\theta}\right)$ **M1 A1**

$f(\theta) = \left(\frac{k\cos^2\theta}{1+\sin\theta}\right) = \frac{k(1-\sin^2\theta)}{1+\sin\theta} = k(1 - \sin\theta)$ **M1 A1)**

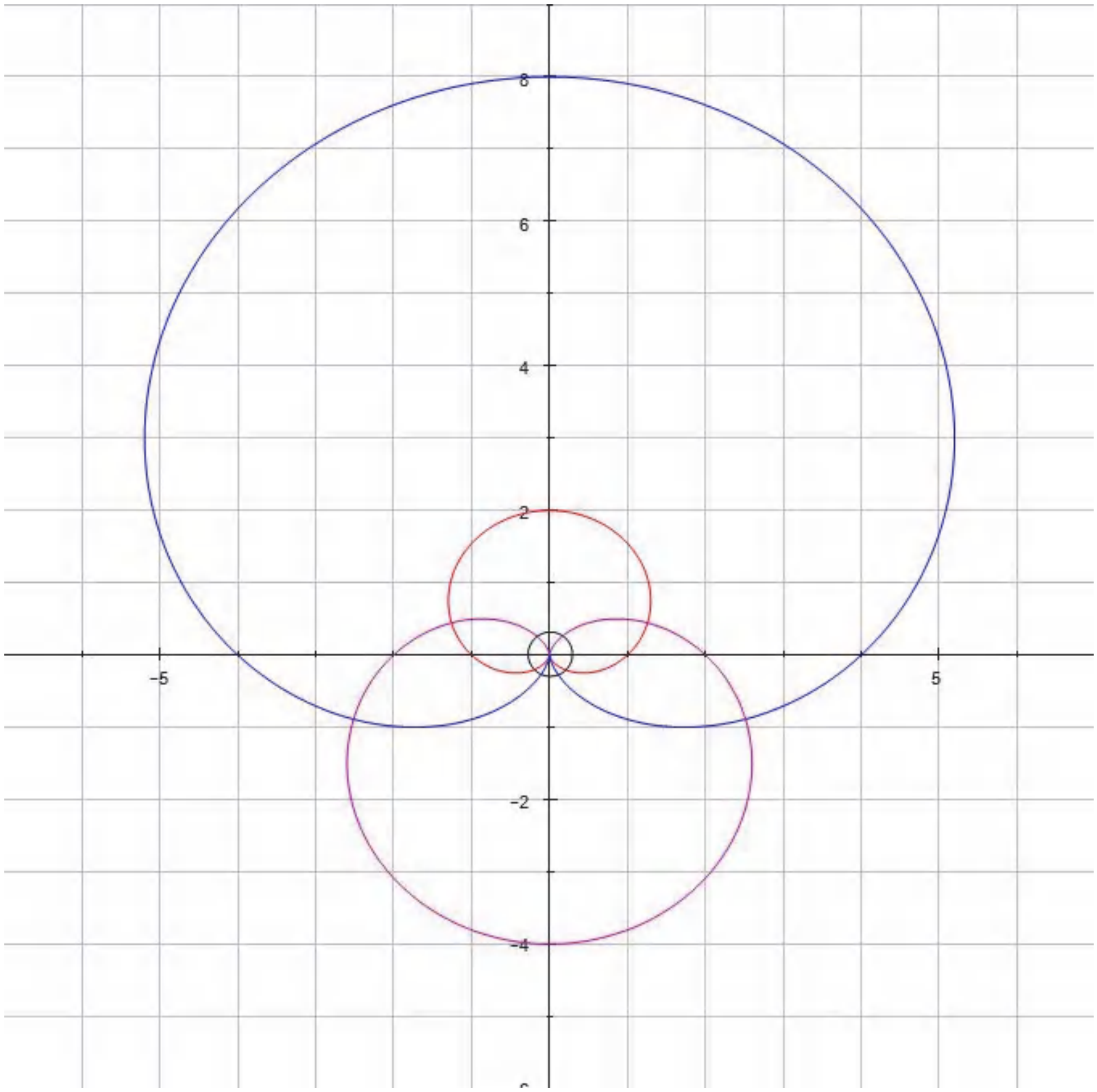
Alternatively, $\frac{f'}{f} = -\frac{(1+\sin\theta)}{\cos\theta} = -\frac{\cos\theta}{(1-\sin\theta)}$ **M1 A1**

$\ln f = \ln\big((1 - \sin\theta)\big) + c = \ln\big(k(1 - \sin\theta)\big)$ **M1**

and hence $f(\theta) = k(1 - \sin\theta)$ **A1**

$r = 4, \theta = -\frac{1}{2}\pi$ so $4 = 2k$ **M1**

Thus $f(\theta) = 2(1 - \sin\theta)$ **A1 (8)**



G1 G1 dG1 G1 G1(5)

Question 6

(i) $T(x) = \int_0^x \frac{1}{1+u^2} du$

Let $u = v^{-1}$, $\frac{du}{dv} = -v^{-2}$

B1

So

$$T(x) = \int_\infty^{x^{-1}} \frac{1}{1+v^{-2}} \times -v^{-2} dv = \int_{x^{-1}}^\infty \frac{1}{v^2+1} dv = \int_0^\infty \frac{1}{1+u^2} du - \int_0^{x^{-1}} \frac{1}{1+u^2} du$$

M1 **M1**

$T(x) = T(\infty) - T(x^{-1})$ **A1* (4)**

(ii) $v = \frac{u+a}{1-au} \Leftrightarrow v - auv = u + a \Leftrightarrow v - u = a(1+uv) \Leftrightarrow a = \frac{v-u}{1+uv}$

M1

$0 = \frac{(1+uv)\left(\frac{dv}{du}-1\right)-(v-u)\left(u\frac{dv}{du}+v\right)}{(1+uv)^2}$ **M1**

$\frac{dv}{du}(1 + uv - uv + u^2) = 1 + uv + v^2 - uv$

$\frac{dv}{du} = \frac{1+v^2}{1+u^2}$ **A1* (3)**

Alternatively,

$$v = \frac{u+a}{1-au} \Leftrightarrow \frac{dv}{du} = \frac{(1-au)+a(u+a)}{(1-au)^2} = \frac{1+a^2}{(1-au)^2} = \frac{(1+a^2)(1+u^2)}{(1-au)^2(1+u^2)}$$

M1

$$= \frac{(1-au)^2+(u+a)^2}{(1-au)^2(1+u^2)} = \frac{1+v^2}{1+u^2}$$

M1 **A1**

$$T(x) = \int_0^x \frac{1}{1+u^2} du = \int_a^{\frac{x+a}{1-ax}} \frac{1}{1+u^2}\frac{1+u^2}{1+v^2} dv = \int_a^{\frac{x+a}{1-ax}} \frac{1}{1+v^2} dv = \int_0^{\frac{x+a}{1-ax}} \frac{1}{1+v^2} dv - \int_0^a \frac{1}{1+v^2} dv$$

M1 **M1**

$T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a)$ **A1* (3)**

As $T(x) = T(\infty) - T(x^{-1})$, $T(a) = T(\infty) - T(a^{-1})$

So

$$T(x^{-1}) = T(\infty) - T(x) = T(\infty) - \left(T\left(\frac{x+a}{1-ax}\right) - T(a)\right) = T(\infty) - \left(T\left(\frac{x+a}{1-ax}\right) - \left(T(\infty) - T(a^{-1})\right)\right)$$

M1 **M1**

Thus

$T(x^{-1}) = 2T(\infty) - T\left(\frac{x+a}{1-ax}\right) - T(a^{-1})$ **A1* (3)**

Let $y = x^{-1}$, $= a^{-1}$, then $x < \frac{1}{a}$ implies $\frac{1}{y} < b$ which is $y > \frac{1}{b}$ **M1**

$T(y) = 2T(\infty) - T\left(\frac{y^{-1}+b^{-1}}{1-b^{-1}y^{-1}}\right) - T(b) = 2T(\infty) - T\left(\frac{b+y}{by-1}\right) - T(b)$ **A1* (2)**

(iii) Using $T(y) = 2T(\infty) - T\left(\frac{b+y}{by-1}\right) - T(b)$ with $y = b = \sqrt{3}$ **M1**

$T(\sqrt{3}) = 2T(\infty) - T\left(\frac{\sqrt{3}+\sqrt{3}}{\sqrt{3}\sqrt{3}-1}\right) - T(\sqrt{3})$

$T(\sqrt{3}) = 2T(\infty) - T(\sqrt{3}) - T(\sqrt{3})$

$3T(\sqrt{3}) = 2T(\infty) \Leftrightarrow T(\sqrt{3}) = \frac{2}{3}T(\infty)$ **A1* (2)**

Using $T(x) = T(\infty) - T(x^{-1})$ with $x = 1$,

$T(1) = T(\infty) - T(1)$ and so $T(1) = \frac{1}{2}T(\infty)$ **B1**

Using $T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a)$ with $x = \sqrt{2} - 1$ and $a = 1$ **M1**

$T(\sqrt{2} - 1\ ) = T\left(\frac{\sqrt{2}-1\ +1}{1-(\sqrt{2}-1\ )}\right) - T(1)$

$T(\sqrt{2} - 1\ ) = T\left(\frac{\sqrt{2}}{2-\sqrt{2}}\right) - T(1) = T\left(\frac{1}{\sqrt{2}-1}\right) - T(1) = T(\sqrt{2} + 1) - T(1)$

Using $T(x) = T(\infty) - T(x^{-1})$, $T(\sqrt{2} + 1\ ) = T(\infty) - T(\sqrt{2} - 1)$

So $T(\sqrt{2} - 1\ ) = T(\infty) - T(\sqrt{2} - 1) - T(1)$

$2T(\sqrt{2} - 1\ ) = T(\infty) - T(1) = T(\infty) - \frac{1}{2}T(\infty)$

$T(\sqrt{2} - 1\ ) = \frac{1}{4}T(\infty)$ **A1* (3)**

Alternatively, using $T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a)$ with $x = a = \sqrt{2} - 1$

$$T(\sqrt{2} - 1\ ) = T\left(\frac{2(\sqrt{2} - 1\ )}{1 - (\sqrt{2} - 1\ )^2}\right) - T(\sqrt{2} - 1\ ) = T\left(\frac{2(\sqrt{2} - 1\ )}{2(\sqrt{2} - 1\ )}\right) - T(\sqrt{2} - 1\ )$$

Therefore $2T(\sqrt{2} - 1\ ) = T(1)$ and so $T(\sqrt{2} - 1\ ) = \frac{1}{2}T(1) = \frac{1}{4}T(\infty)$

Question 7

$$\frac{\frac{a^2(1-t^2)^2}{(1+t^2)^2}}{a^2}+\frac{\frac{4b^2t^2}{(1+t^2)^2}}{b^2}=\frac{(1-t^2)^2+4t^2}{(1+t^2)^2}=\frac{1-2t^2+t^4+4t^2}{1+2t^2+t^4}=1$$ **B1 (1)**

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}+\frac{2y\frac{dy}{dx}}{b^2}=0$ **M1**

$\frac{dy}{dx}=-\frac{b^2x}{a^2y}=-\frac{b^2a(1-t^2)(1+t^2)}{a^2(1+t^2)2bt}=-\frac{b(1-t^2)}{2at}$ **M1 A1**

So L is $y-\frac{2bt}{(1+t^2)}=-\frac{b(1-t^2)}{2at}\left(x-\frac{a(1-t^2)}{(1+t^2)}\right)$ **M1**

$2at(1+t^2)y-4abt^2=-bx(1-t^2)(1+t^2)+ab(1-t^2)^2$

$2at(1+t^2)y+bx(1-t^2)(1+t^2)=ab(1-t^2)^2+4abt^2=ab(1+t^2)^2$

Thus $2aty+bx(1-t^2)=ab(1+t^2)$ **M1**

and as (X,Y) lies on this line $2atY+bX(1-t^2)=ab(1+t^2)$

$0=(a+X)bt^2-2atY+b(a-X)$ **A1* (6)**

For there to be two distinct lines, there need to be two values of t .

So the discriminant must be positive, $(-2aY)^2-4(a+X)bb(a-X)>0$ **M1**

$4a^2Y^2>4b^2(a^2-X^2)$

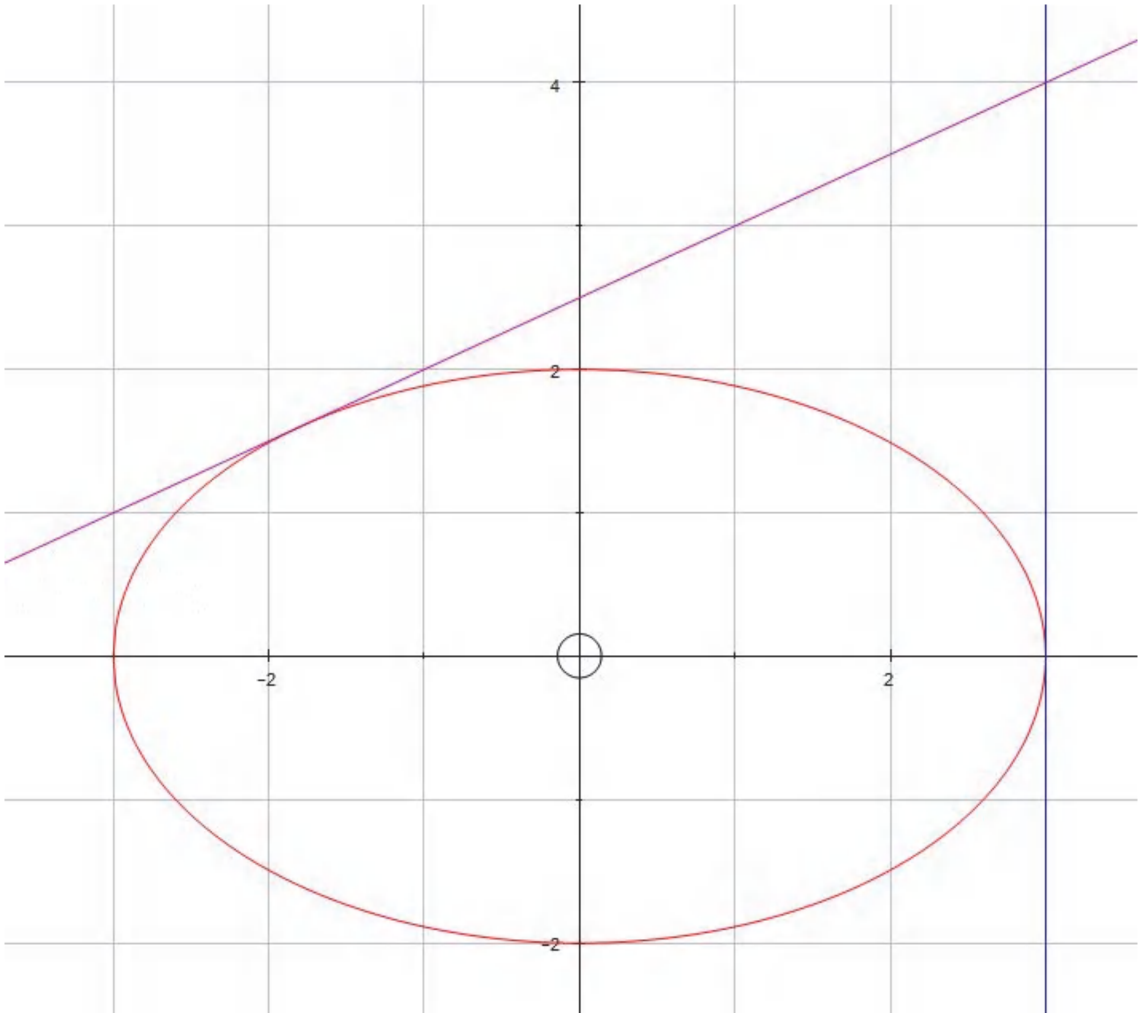
$a^2Y^2>(a^2-X^2)b^2$ **A1***

$\frac{Y^2}{b^2}>1-\frac{X^2}{a^2}$

$\frac{X^2}{a^2}+\frac{Y^2}{b^2}>1$ so (X,Y) lies outside the ellipse. **B1 (3)**

However, if $X^2=a^2$, $=\pm a$, one tangent is at $t=0$ or $t=\infty$, a vertical line. **E1**

If $\frac{X^2}{a^2}+\frac{Y^2}{b^2}>1$, then $Y\neq 0$. **E1**



G1 (3)

(ii) p and q are the roots of $0 = (a + X)bt^2 - 2atY + b(a - X)$

So $p + q = \frac{2aY}{(a+X)b}$ and $pq = \frac{b(a-X)}{(a+X)b}$ **M1**

Thus $(a + X)pq = a - X$ and $(a + X)(p + q)b = 2aY$ **A1 A1 (3)**

Without loss of generality $(0, y_1)$ lies on $(a + x)bp^2 - 2apy + b(a - x) = 0$

and $(0, y_2)$ lies on $(a + x)bq^2 - 2aqy + b(a - x) = 0$

So $abp^2 - 2apy_1 + ab = 0$, that is $bp^2 - 2py_1 + b = 0$ **M1**

and $bq^2 - 2qy_2 + b = 0$

As $y_1 + y_2 = 2b$, $\frac{bp^2+b}{2p} + \frac{bq^2+b}{2q} = 2b$ **M1**

$$\frac{p^2+1}{p} + \frac{q^2+1}{q} = 4$$

$$p + q + \frac{p+q}{pq} = 4$$

$$\frac{2aY}{(a+X)b} + \frac{\frac{2aY}{(a+X)b}}{\frac{a-X}{a+X}} = 4$$ **M1**

$$\frac{2aY}{a+X} + \frac{2aY}{a-X} = 4b$$

$2aY(a - X + a + X) = 4(a - X)(a + X)b$

$4a^2Y = 4(a^2 - X^2)b$

$\frac{Y}{b} = 1 - \frac{X^2}{a^2}$

$\frac{X^2}{a^2} + \frac{Y}{b} = 1$ **A1* (4)**

Question 8

$$\sum_{m=1}^{n} a_m(b_{m+1}-b_m)+\sum_{m=1}^{n} b_{m+1}(a_{m+1}-a_m)$$
$$=\sum_{m=1}^{n}(a_m b_{m+1}-a_m b_m+b_{m+1}a_{m+1}-b_{m+1}a_m)$$

M1

$$=\sum_{m=1}^{n}(-a_m b_m+b_{m+1}a_{m+1})=a_{n+1}b_{n+1}-a_1b_1$$

M1

Hence,

$$\sum_{m=1}^{n} a_m(b_{m+1}-b_m)=a_{n+1}b_{n+1}-a_1b_1-\sum_{m=1}^{n} b_{m+1}(a_{m+1}-a_m)$$

A1* (3)

(i) Let $a_m=1$ (or any constant) and $b_m=\sin mx$, **M1**

then

$$\sum_{m=1}^{n}(\sin(m+1)x-\sin mx)=\sin(n+1)x-\sin x-\sum_{m=1}^{n}\sin(m+1)x\ \ (1-1)$$

M1 A1

So

$$\sum_{m=1}^{n} 2\cos\left(m+\frac{1}{2}\right)x\ \sin\frac{1}{2}x=\ (\sin(n+1)x-\sin x)$$

M1 A1

and therefore

$$\sum_{m=1}^{n}\cos\left(m+\frac{1}{2}\right)x=\frac{1}{2}(\sin(n+1)x-\sin x)\csc\frac{1}{2}x$$

A1* (6)

(ii) Let $a_m=m$ and $b_m=\sin(m-1)x-\sin mx$, **M1**

then

$$b_{m+1}-b_m=(\sin mx-\sin(m+1)x)-(\sin(m-1)x-\sin mx)$$
$$=-2\cos\left(m+\frac{1}{2}\right)x\ \sin\frac{1}{2}x+\ 2\cos\left(m-\frac{1}{2}\right)x\ \sin\frac{1}{2}x$$

M1 A1

$= 4\sin mx\ \sin\frac{1}{2}x\sin\frac{1}{2}x$ **M1 A1**

Thus, using the stem

$$\sum_{m=1}^{n} m\times 4\sin mx\ \sin^2\frac{1}{2}x$$
$$= (n+1)(\sin nx - \sin(n+1)x) - 1\times(\sin(0\times x) - \sin x)$$
$$- \sum_{m=1}^{n}(\sin mx - \sin(m+1)x)$$

M1 A1

So

$$4\sin^2\frac{1}{2}x\sum_{m=1}^{n} m\sin mx\ = (n+1)(\sin nx - \sin(n+1)x) + \sin x - \sin x + \sin(n+1)x$$

M1 A1

$$4\sin^2\frac{1}{2}x\sum_{m=1}^{n} m\sin mx\ = (n+1)\sin nx - n\sin(n+1)x$$

Thus

$$\sum_{m=1}^{n} m\sin mx\ = (p\sin nx + q\sin(n+1)x)\csc^2\frac{1}{2}x$$

where

$$p = -\frac{1}{4}n$$

A1

and

$$q = \frac{1}{4}(n+1)$$

A1 (11)

Alternatively, let $a_m = m$ and $b_m = \cos\left(m-\frac{1}{2}\right)x$, using stem, **M1**

$$\sum_{m=1}^{n} m\left(\cos\left(m+\frac{1}{2}\right)x - \cos\left(m-\frac{1}{2}\right)x\right)$$
$$= (n+1)\cos\left(n+\frac{1}{2}\right)x - \cos\frac{1}{2}x - \sum_{m=1}^{n}\cos\left(m+\frac{1}{2}\right)x$$

M1 A1

So,

$$\sum_{m=1}^{n} -2m \sin mx \ \sin\frac{1}{2}x$$

$$= (n+1)\cos\left(n+\frac{1}{2}\right)x - \cos\frac{1}{2}x - \frac{1}{2}(\sin(n+1)x - \sin x)\csc\frac{1}{2}x$$

M1 A1

$$= \csc\frac{1}{2}x\left((n+1)\cos\left(n+\frac{1}{2}\right)x\sin\frac{1}{2}x - \sin\frac{1}{2}x\cos\frac{1}{2}x - \frac{1}{2}(\sin(n+1)x - \sin x)\right)$$

M1 A1

$$= \frac{1}{2}\csc\frac{1}{2}x\left(2(n+1)\cos\left(n+\frac{1}{2}\right)x\sin\frac{1}{2}x - 2\sin\frac{1}{2}x\cos\frac{1}{2}x - (\sin(n+1)x - \sin x)\right)$$

$$= \frac{1}{2}\csc\frac{1}{2}x\left((n+1)(\sin(n+1)x - \sin nx) - \sin x - \sin(n+1)x + \sin x\right)$$

M1 A1

$$= \frac{1}{2}\csc\frac{1}{2}x\,(n\sin(n+1)x - (n+1)\sin nx)$$

giving result as before.

Question 9

For A, $mg - Z = m\ddot{y}$ and for B, $Z = 2m\ddot{x}$ where Z is tension. **M1 A1 A1**

Adding, $\ddot{y} + 2\ddot{x} = g$ **M1**

Integrating with respect to time, $\dot{y} + 2\dot{x} = gt + c$

Initially, $t = 0$, $\dot{x} = 0$, $\dot{y} = 0 \Rightarrow c = 0$

Integrating with respect to time, $y + 2x = \frac{1}{2}gt^2 + c'$ **M1 M1**

Initially, $t = 0$, $x = 0$, $y = 0 \Rightarrow c' = 0$

So $y + 2x = \frac{1}{2}gt^2$ **A1* (7)**

When $x = a$, $t = T = \sqrt{\frac{6a}{g}}$ so $y = a$ **M1 A1**

Conserving energy, at time T we have shown there is no elastic potential energy, so

$$0 = \frac{1}{2}2m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mga$$

M1 A1 A1 A1 (6)

That is

$$2\dot{x}^2 + \dot{y}^2 = 2ga$$

B1

But also $\dot{y} + 2\dot{x} = gT$ and so $\dot{y} + 2\dot{x} = \sqrt{6ga}$ **M1 A1**

Thus $2\dot{x}^2 + \left(\sqrt{6ga} - 2\dot{x}\right)^2 = 2ga$ **M1 A1**

$$6\dot{x}^2 - 4\dot{x}\sqrt{6ga} + 4ga = 0$$

$$\dot{x}^2 - 2\dot{x}\sqrt{\frac{2ga}{3}} + \frac{2ga}{3} = 0$$

$$\left(\dot{x} - \sqrt{\frac{2ga}{3}}\right)^2 = 0$$

M1

and so $\dot{x} = \sqrt{\frac{2ga}{3}}$ **A1* (7)**

Alternatively,

$$Z = \frac{\lambda(y - x)}{a}$$

M1

Subtracting,

$$2mg - 3Z = 2m(\ddot{y} - \ddot{x})$$

$$\ddot{y} - \ddot{x} = -\frac{3\lambda(y - x)}{2ma} + g$$

M1

So,

$$y - x = \frac{2mga}{3\lambda}(1 - \cos\omega t)$$

M1

where

$$\omega^2 = \frac{3\lambda}{2ma}$$

As $y + 2x = \frac{1}{2}gt^2$, $3x = \frac{1}{2}gt^2 - \frac{2mga}{3\lambda}(1 - \cos\omega t)$ **M1**

When $x = a$, $t = T = \sqrt{\frac{6a}{g}}$

so $3a = 3a - \frac{2mga}{3\lambda}\left(1 - \cos\omega\sqrt{\frac{6a}{g}}\right)$ and thus $\frac{3\lambda}{2ma}\frac{6a}{g} = 4n^2\pi^2$, $\lambda = \frac{4n^2\pi^2 mg}{9}$ **M1**

$$3\dot{x} = gt - \frac{2mga\omega\sin\omega t}{3\lambda} = g\sqrt{\frac{6a}{g}} - 0$$

$$\dot{x} = \sqrt{\frac{2ga}{3}}$$

M1 A1* (7)

Question 10

Moment of inertia of PQ about axis through P is $\frac{1}{3}m(3a)^2 = 3ma^2$ **B1**

Conserving energy, $0 = \frac{1}{2}3ma^2\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - mg\frac{3}{2}a\sin\theta - mgl\sin\theta$ **M1 A1 A1 A1**

Thus $(3a^2 + l^2)\dot{\theta}^2 = g(3a + 2l)\sin\theta$ **A1* (6)**

Differentiating with respect to time,

$$2(3a^2 + l^2)\dot{\theta}\ddot{\theta} = g(3a + 2l)\cos\theta\dot{\theta}$$

M1

So

$$2(3a^2 + l^2)\ddot{\theta} = g(3a + 2l)\cos\theta$$

A1 (2)

Alternatively, taking moments about axis through P

$$m(3a^2 + l^2)\ddot{\theta} = mg\left(\frac{3}{2}a + l\right)\cos\theta$$

M1

So

$$2(3a^2 + l^2)\ddot{\theta} = g(3a + 2l)\cos\theta$$

A1 (2)

Resolving perpendicular to the rod for the particle,

$$mg\cos\theta - R = ml\ddot{\theta}$$

M1 A1

Thus

$$R = mg\cos\theta - ml\ddot{\theta} = mg\cos\theta\left(1 - \frac{l(3a + 2l)}{2(3a^2 + l^2)}\right)$$

M1 A1

$$1 - \frac{l(3a + 2l)}{2(3a^2 + l^2)} = \frac{6a^2 + 2l^2 - 3al - 2l^2}{2(3a^2 + l^2)} = \frac{3a(2a - l)}{2(3a^2 + l^2)} > 0$$

because $l < 2a$ **A1 (5)**

Resolving along the rod towards P for the particle,

$$F - mg\sin\theta = ml\dot{\theta}^2$$

M1 A1

Thus

$$F = mg\sin\theta + ml\dot{\theta}^2 = mg\sin\theta\left(1 + \frac{l(3a+2l)}{(3a^2+l^2)}\right) = mg\sin\theta\left(\frac{3(a^2+al+l^2)}{(3a^2+l^2)}\right)$$

M1

On the point of slipping $F = \mu R$, so **B1**

$$mg\sin\theta\left(\frac{3(a^2+al+l^2)}{(3a^2+l^2)}\right) = \mu mg\cos\theta\left(\frac{3a(2a-l)}{2(3a^2+l^2)}\right)$$

Thus

$$\tan\theta = \frac{\mu a(2a-l)}{2(a^2+al+l^2)}$$

A1* (5)

At the instant of release, the equation of rotational motion for the rod ignoring the particle is

$$mg\frac{3a}{2} = 3ma^2\ddot{\theta}$$

and thus

$$\ddot{\theta} = \frac{g}{2a}$$

M1

Therefore the acceleration of the point on the rod where the particle rests equals $l\ddot{\theta} = \frac{lg}{2a} > g$ if $l > 2a$, and so the rod drops away from the particle faster than the particle accelerates and the particle immediately loses contact. **A1 (2)**

(Alternatively, for particle to accelerate with rod from previous working $R < 0$, **M1** meaning that it would have to be attached to so accelerate, and as it is only placed on the rod, this cannot happen.) **A1 (2)**

Question 11

(i) Conserving (linear) momentum

$$Mu - nmv = 0$$

M1

$$u = \frac{nmv}{M}$$

A1

$$K = \frac{1}{2}Mu^2 + n \times \frac{1}{2}mv^2 = \frac{1}{2}M\left(\frac{nmv}{M}\right)^2 + \frac{1}{2}nmv^2 = \frac{1}{2}nmv^2\left(\frac{nm}{M} + 1\right)$$

M1 **M1** **A1* (5)**

as required.

(ii) Conserving momentum before and after r th gun fired

$$\big(M + \big(n - (r-1)\big)m\big)u_{r-1} = (M + (n-r)m)u_r - m(v - u_{r-1})$$

M1 A1

Therefore

$$(M + (n-r)m)(u_r - u_{r-1}) = mv$$

M1

and so

$$u_r - u_{r-1} = \frac{mv}{M + (n-r)m}$$

A1* (4)

Summing this result for $r = 1$ to $r = n$,

$$u_n - u_0 = \frac{mv}{M + (n-1)m} + \frac{mv}{M + (n-2)m} + \frac{mv}{M + (n-3)m} + \cdots + \frac{mv}{M + (n-n)m}$$

M1

Because

$$0 \le n - r \le n - 1$$

$$M \le M + (n-r)m \le M + (n-1)m$$

$$\frac{mv}{M + (n-1)m} \le \frac{mv}{M + (n-r)m} \le \frac{mv}{M}$$

with equality only for the term $r = n$

Thus

$$\frac{mv}{M + (n-1)m} + \frac{mv}{M + (n-2)m} + \frac{mv}{M + (n-3)m} + \cdots + \frac{mv}{M + (n-n)m} < \frac{nmv}{M}$$

E1

As $u_0 = 0$, $u_n < \frac{nmv}{M} = u$ **A1* (3)**

(iii) Considering the energy of the truck and the $(n-(r-1))$ projectiles before and after the r^{th} projectile is fired (the other $(r-1)$ already fired do not change their kinetic energy at this time),

$$K_r - K_{r-1} = \frac{1}{2}(M+(n-r)m)u_r{}^2 + \frac{1}{2}m(v-u_{r-1})^2 - \frac{1}{2}\big(M+(n-(r-1))m\big)u_{r-1}{}^2$$

M1 A1

$$= \frac{1}{2}(M+(n-r)m)(u_r{}^2 - u_{r-1}{}^2) + \frac{1}{2}m(v-u_{r-1})^2 - \frac{1}{2}mu_{r-1}{}^2$$

$$= \frac{1}{2}(M+(n-r)m)(u_r - u_{r-1})(u_r + u_{r-1}) + \frac{1}{2}mv^2 - mvu_{r-1}$$

$$= \frac{1}{2}mv(u_r + u_{r-1}) + \frac{1}{2}mv^2 - mvu_{r-1}$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}mv(u_r - u_{r-1})$$

M1

Summing this result for $r=1$ to $r=n$,

$$K_n - K_0 = \frac{1}{2}nmv^2 + \frac{1}{2}mv(u_n - u_0)$$

M1

So

$$K_n = \frac{1}{2}nmv^2 + \frac{1}{2}mvu_n$$

A1* (5)

Now

$$u_n < \frac{nmv}{M}$$

so

$$\frac{1}{2}mvu_n < \frac{1}{2}\frac{nm^2v^2}{M}$$

M1

and thus

$$K_n = \frac{1}{2}nmv^2 + \frac{1}{2}mvu_n < \frac{1}{2}nmv^2 + \frac{1}{2}\frac{nm^2v^2}{M} = \frac{1}{2}nmv^2\left(1+\frac{m}{M}\right) < \frac{1}{2}nmv^2\left(\frac{nm}{M}+1\right)$$
$$= K$$

M1

as $n > 1$ **E1 (3)**

Question 12

(i)

$$\sum_{y=1}^{n}\sum_{x=1}^{n} P(X = x, Y = y) = 1$$

$$\sum_{y=1}^{n}\sum_{x=1}^{n} k(x + y) = 1$$

M1

$$k\sum_{y=1}^{n}\left(\frac{1}{2}n(n+1) + ny\right) = 1$$

M1 A1

$$k\left(\frac{1}{2}n^2(n+1) + \frac{1}{2}n^2(n+1)\right) = 1$$

M1

Therefore,

$$k = \frac{1}{n^2(n+1)}$$

A1 (5)

$$P(X = x) = \sum_{y=1}^{n} k(x + y) = k\left(nx + \frac{1}{2}n(n+1)\right) = \frac{\big(2nx + n(n+1)\big)}{2n^2(n+1)} = \frac{n+1+2x}{2n(n+1)}$$

M1 A1 (2)

$$P(Y = y) = \frac{n+1+2y}{2n(n+1)}$$

B1

For X and Y to be independent, $P(X = x, Y = y) = P(X = x) \times P(Y = y)$ **M1**

So

$$\frac{n+1+2x}{2n(n+1)} \times \frac{n+1+2y}{2n(n+1)} = \frac{(x+y)}{n^2(n+1)}$$

M1

$$(n+1+2x)(n+1+2y) = 4(n+1)(x+y)$$

$$(n+1)^2 - 2(n+1)(x+y) + 4xy = 0$$

$$\big((n+1) - (x+y)\big)^2 - (x-y)^2 = 0$$

M1

which does not happen for e.g. $x = n$, $y = 1$. (Many equally valid examples possible.)

X and Y are not independent. **E1 (5)**

(ii)

$$E(XY) = \sum_{y=1}^{n}\sum_{x=1}^{n} kxy(x+y) = k\sum_{y=1}^{n}\left(y\frac{n(n+1)(2n+1)}{6} + y^2\frac{n(n+1)}{2}\right)$$

M1

$$= k\frac{n^2(n+1)^2(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

M1 A1 (3)

$$E(X) = E(Y) = \sum_{x=1}^{n} x\frac{n+1+2x}{2n(n+1)} = \frac{\frac{1}{2}n(n+1)^2 + \frac{2}{6}n(n+1)(2n+1)}{2n(n+1)}$$

M1 A1

$$= \frac{(n+1)}{4} + \frac{(2n+1)}{6} = \frac{(7n+5)}{12}$$

A1 (3)

Thus

$$Cov(X,Y) = \frac{(n+1)(2n+1)}{6} - \left(\frac{(7n+5)}{12}\right)^2 = \frac{-n^2+2n-1}{144} = \frac{-(n-1)^2}{144} < 0$$

M1 **E1 (2)**

Question 13

$$V(x) = E((X-x)^2) = E(X^2) - 2xE(X) + x^2 = \sigma^2 + \mu^2 - 2x\mu + x^2 = \sigma^2 + (x-\mu)^2$$

M1 M1 M1 A1 (4)

$$E(Y) = E\big(V(X)\big) = E(\sigma^2 + (X-\mu)^2) = \sigma^2 + \sigma^2 = 2\sigma^2$$

M1 A1* (2)

If $X \sim U(0,1)$, then $\mu = \frac{1}{2}$ and $\sigma^2 = \frac{1}{12}$, so $V(x) = \frac{1}{12} + \left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{3}$

B1 B1 M1 A1 (4)

$$Y = V(X) = X^2 - X + \frac{1}{3} = \frac{1}{12} + \left(X - \frac{1}{2}\right)^2$$

$$Y \in \left[\frac{1}{12}, \frac{1}{3}\right]$$

$$P(Y < y) = P\left(\frac{1}{12} + \left(X - \frac{1}{2}\right)^2 < y\right) = P\left(\frac{1}{2} - \sqrt{y - \frac{1}{12}} < X < \frac{1}{2} + \sqrt{y - \frac{1}{12}}\right)$$

$$= 2\sqrt{y - \frac{1}{12}}$$

M1 M1 A1

$f(y) = \frac{d}{dy}\big(F(y)\big) = \frac{d}{dy}\left(2\sqrt{y - \frac{1}{12}}\right) = \left(y - \frac{1}{12}\right)^{-\frac{1}{2}}$, $\frac{1}{12} \leq y \leq \frac{1}{3}$ and 0 otherwise.

M1 A1 A1 (6)

$$E(Y) = \int_{\frac{1}{12}}^{\frac{1}{3}} y\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} dy = \int_{\frac{1}{12}}^{\frac{1}{3}} \left(y - \frac{1}{12}\right)\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} + \frac{1}{12}\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} dy$$

M1 M1

$$= \left[\frac{2}{3}\left(y - \frac{1}{12}\right)^{\frac{3}{2}} + \frac{1}{6}\left(y - \frac{1}{12}\right)^{\frac{1}{2}}\right]_{\frac{1}{12}}^{\frac{1}{3}} = \frac{1}{12} + \frac{1}{12} = 2 \times \frac{1}{12}$$

M1 A1 (4)

as required.

Alternatively, for final integral,

let $u^2 = y - \frac{1}{12}$,

$$E(Y) = \int_{\frac{1}{12}}^{\frac{1}{3}} y\left(y-\frac{1}{12}\right)^{-\frac{1}{2}} dy = \int_{0}^{\frac{1}{2}} \frac{u^2+\frac{1}{12}}{u}\, 2u du = \left[\frac{2}{3}u^3+\frac{1}{6}u\right]_0^{\frac{1}{2}} = 2\times\frac{1}{12}$$

M1 **M1** **M1** **A1 (4)**

or further

let $u = y - \frac{1}{12}$,

$$E(Y) = \int_{\frac{1}{12}}^{\frac{1}{3}} y\left(y-\frac{1}{12}\right)^{-\frac{1}{2}} dy = \int_{0}^{\frac{1}{4}} \frac{u+\frac{1}{12}}{u^{\frac{1}{2}}}\, du = \left[\frac{2}{3}u^{\frac{3}{2}}+\frac{1}{6}u^{\frac{1}{2}}\right]_0^{\frac{1}{4}} = 2\times\frac{1}{12}$$

M1 **M1** **M1** **A1 (4)**

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