2019AIME II 真题

Problem 1

Two different points, C and D, lie on the same side of line AB so that $\triangle ABC$ and $\triangle BAD$ are congruent with $AB = 9, BC = AD = 10$, and $CA = DB = 17$. The intersection of these two triangular regions has area $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 2

Lily pads $1, 2, 3, \ldots$ lie in a row on a pond. A frog makes a sequence of jumps starting on pad 1. From any pad k the frog jumps to either pad $k + 1$ or pad $k + 2$ chosen randomly with probability $\frac{1}{2}$ and independently of other jumps. The probability that the frog visits pad 7 is $\frac{p}{q}$, where $p$and q are relatively prime positive integers. Find $p + q$.

Problem 3

Find the number of 7-tuples of positive integers (a, b, c, d, e, f, g) that satisfy the following system of equations:$abc = 70cde = 71efg = 72.$

Problem 4

A standard six-sided fair die is rolled four times. The probability that the product of all four numbers rolled is a perfect square is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 5

Four ambassadors and one advisor for each of them are to be seated at a round table with 12 chairs numbered in order 1 to 12. Each ambassador must sit in an even-numbered chair. Each advisor must sit in a chair adjacent to his or her ambassador. There are N ways for the 8people to be seated at the table under these conditions. Find the remainder when N is divided by 1000.

Problem 6

In a Martian civilization, all logarithms whose bases are not specified as assumed to be base b, for some fixed $b \geq 2$. A Martian student writes down$3\log(\sqrt{x}\log x) = 56\log_{\log x}(x) = 54$and finds that this system of equations has a single real number solution $x > 1$. Find b.

Problem 7

Triangle ABC has side lengths $AB = 120, BC = 220$, and $AC = 180$. Lines ℓ_A, ℓ_B, and ℓ_C are drawn parallel to $\overline{BC}, \overline{AC}$, and $\overline{AB}$, respectively, such that the intersections of ℓ_A, ℓ_B, and ℓ_C with the interior of $\triangle ABC$ are segments of lengths $55, 45$, and 15, respectively. Find the perimeter of the triangle whose sides lie on lines ℓ_A, ℓ_B, and ℓ_C.

Problem 8

The polynomial $f(z) = az^{2018} + bz^{2017} + cz^{2016}$ has real coefficients not exceeding 2019, and $f\left(\frac{1+\sqrt{3}i}{2}\right) = 2015 + 2019\sqrt{3}i$. Find the remainder when $f(1)$ is divided by 1000.

Problem 9

Call a positive integer n k*-pretty* if n has exactly k positive divisors and n is divisible by k. For example, 18 is 6-pretty. Let S be the sum of the positive integers less than 2019 that are 20-pretty. Find $\frac{S}{20}$.

Problem 10

There is a unique angle θ between 0° and 90° such that for nonnegative integers n, the value of $\tan(2^n\theta)$ is positive when n is a multiple of 3, and negative otherwise. The degree measure of θ is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Problem 11

Triangle ABC has side lengths $AB=7, BC=8,$ and $CA=9.$ Circle ω_1 passes through B and is tangent to line AC at $A.$ Circle $\omega_2$passes through C and is tangent to line AB at $A.$ Let K be the intersection of circles ω_1 and ω_2 not equal to $A.$ Then $AK=\frac{m}{n},$ where m and n are relatively prime positive integers. Find $m+n.$

Problem 12

For $n\geq 1$ call a finite sequence $(a_1,a_2,\ldots,a_n)$ of positive integers *progressive* if $a_i<a_{i+1}$ and a_i divides a_{i+1} for $1\leq i\leq n-1$. Find the number of progressive sequences such that the sum of the terms in the sequence is equal to $360.$

Problem 13

Regular octagon $A_1A_2A_3A_4A_5A_6A_7A_8$ is inscribed in a circle of area $1.$ Point P lies inside the circle so that the region bounded by $\overline{PA_1},\overline{PA_2},$ and the minor arc $\widehat{A_1A_2}$ of the circle has area $\frac{1}{7},$ while the region bounded by $\overline{PA_3},\overline{PA_4},$ and the minor arc $\widehat{A_3A_4}$ of the circle has area $\frac{1}{9}.$ There is a positive integer n such that the area of the region bounded by $\overline{PA_6},\overline{PA_7},$ and the minor arc $\widehat{A_6A_7}$ of the circle is equal to $\frac{1}{8}-\frac{\sqrt2}{n}.$ Find $n.$

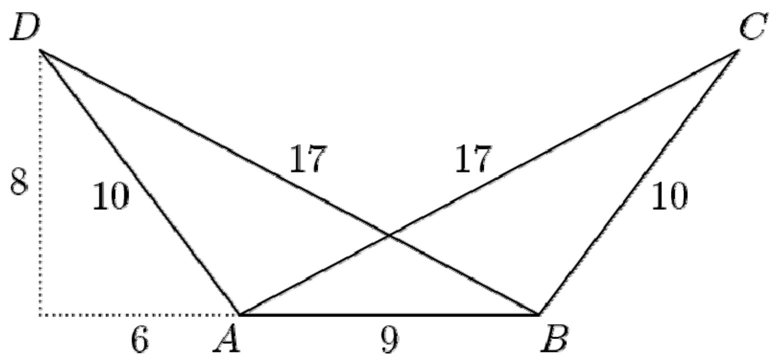
Problem 14

Find the sum of all positive integers n such that, given an unlimited supply of stamps of denominations $5,n,$ and $n+1$ cents, 91 cents is the greatest postage that cannot be formed.

Problem 15

In acute triangle ABC points P and Q are the feet of the perpendiculars from C to $\overline{AB}$ and from B to $\overline{AC}$, respectively. Line PQ intersects the circumcircle of $\triangle ABC$ in two distinct points, X and Y.
Suppose $XP=10$, $PQ=25$, and $QY=15$. The value of $AB\cdot AC$ can be written in the form $m\sqrt n$ where m and n are positive relatively prime integers. Find $m+n$.

参考答案(部分)



1. - Diagram by Brendanb4321Extend AB to form a right triangle with legs 6 and 8 such that AD is the hypotenuse and connect the points CD so that you have a rectangle. The base CD of the rectangle will be $9+6+6=21$. Now, let E be the intersection of BD and AC. This means that $\triangle ABE$ and $\triangle DCE$are with ratio $\frac{21}{9}=\frac{7}{3}$. Set up a proportion, knowing that the two heights add up to 8. We will let y be the height from E to DC, and x be the height of $\triangle ABE$.$\frac{7}{3}=\frac{y}{x}\frac{7}{3}=\frac{8-x}{x}7x=24-3x10x=24^{x}=\frac{12}{5}$ This means that the area is $A=\frac{1}{2}(9)(\frac{12}{5})=\frac{54}{5}$. This gets us $54+5=\boxed{059}$.

-Solution by the Math Wizard, Number Magician of the Second Order, Head of the Council of the Geometers

2.Let P_n be the probability the frog visits pad 7 starting from pad n. Then $P_7 = 1$, $P_6 = \frac{1}{2}$, and $P_n = \frac{1}{2}(P_{n+1} + P_{n+2})$ for all integers $1 \leq n \leq 5$. Working our way down, we find $P_5 = \frac{3}{4}P_4 = \frac{5}{8}P_3 = \frac{11}{16}P_2 = \frac{21}{32}P_1 = \frac{43}{64}43 + 64 = \boxed{107}$.

3.As 71 is prime, c, d, and e must be 1, 1, and 71 (up to ordering). However, since c and e are divisors of 70 and 72 respectively, the only possibility is $(c, d, e) = (1, 71, 1)$. Now we are left with finding the number of solutions (a, b, f, g) satisfying $ab = 70$ and $fg = 72$, which separates easily into two subproblems. The number of positive integer solutions to $ab = 70$ simply equals the number of divisors of 70 (as we can choose a divisor for a, which uniquely determines b). As $70 = 2^1 \cdot 5^1 \cdot 7^1$, we have $d(70) = (1+1)(1+1)(1+1) = 8$ solutions. Similarly, $72 = 2^3 \cdot 3^2$, so $d(72) = 4 \times 3 = 12$.Then the answer is simply $8 \times 12 = \boxed{096}$.

4.Notice that, other than the number 5, the remaining numbers 1, 2, 3, 4, 6 are only divisible by 2 and/or 3. We can do some cases on the number of 5's rolled (note that there are $6^4 = 1296$ outcomes).Case 1 (easy): Four 5's are rolled. This has probability $\frac{1}{6^4}$ of occurring.

Case 2: Two 5's are rolled.

Case 3: No 5's are rolled.

To find the number of outcomes for the latter two cases, we will use recursion. Consider a 5-sided die with faces numbered 1, 2, 3, 4, 6. For $n \geq 1$, let a_n equal the number of outcomes after rolling the die n times, with the property that the product is a square. Thus, $a_1 = 2$ as 1 and 4 are the only possibilities.

To find a_{n+1} given a_n (where $n \geq 1$), we observe that if the first n rolls multiply to a perfect square, then the last roll must be 1 or 4. This gives $2a_n$ outcomes. Otherwise, the first n rolls do not multiply to a perfect square ($5^n - a_n$ outcomes). In this case, we claim that the last roll is uniquely determined (either 2, 3, or 6). If the product of the first n rolls is $2^x 3^y$ where x and y are not both even, then we observe that if x and $y$are both odd, then the last roll must be 6; if only x is odd, the last roll must be 2, and if only y is odd, the last roll must be 3. Thus, we have $5^n - a_n$ outcomes in this case, and $a_{n+1} = 2a_n + (5^n - a_n) = 5^n + a_n$.

Computing a_2, a_3, a_4 gives $a_2 = 7$, $a_3 = 32$, and $a_4 = 157$. Thus for Case 3, there are 157 outcomes. For case 2, we multiply by $\binom{4}{2} = 6$ to distribute the two 5's among four rolls. Thus the probability is

$$\frac{1 + 6 \cdot 7 + 157}{6^4} = \frac{200}{6^4} = \frac{25}{162} \implies m + n = \boxed{187}$$

5.There are 4 ambassadors and there are 6 seats for them. So we consider the position of the blank seats. There are 15 kinds of versions: If the two seats are adjacent to each other, there are 6 options, and the ambassadors are sitting in four adjacent seats, and there are five seats that their advisors can sit. Choose any of them and the advisors' seats are fixed, so there are 5 kinds of solutions for the advisors to sit. And that's a $6 \cdot 5$ if we don't consider the order of the ambassadors. We can also get that if the blank seats are opposite, it will be $3 \cdot 9$, if they are not adjacent and not opposite, it will be $6 \cdot 8$. So the total is $24 \cdot (6 \cdot 5 + 6 \cdot 8 + 3 \cdot 9) = 2520$ And the remainder is $\boxed{520}$

6.From the first equation we have that $\log(\sqrt{x}\log x) = \frac{56}{3}$, so $\log(\sqrt{x}) + \log(\log x) = \frac{1}{2}\log x + \log(\log x) = \frac{56}{3}$. From the second equation we have that $x = (\log x)^{54}$, so now set $\log x = a$ and $x = b^a$. Substituting, we have that $b^a = a^{54}$, so $b = a^{\frac{54}{a}}$. We also have that $\frac{1}{2}a + \log_{a^{\frac{54}{a}}} a = \frac{56}{3}$, so $\frac{1}{2}a + \frac{1}{54}a = \frac{56}{3}$. This means that $\frac{14}{27}a = \frac{56}{3}$, so $a = 36$, and $b = 36^{\frac{54}{36}} = 36^{\frac{3}{2}} = \boxed{216}$.

7.Let the points of intersection of ℓ_a, ℓ_b, ℓ_c with $\triangle ABC$ divide the sides into consecutive segments $BD, DE, EC, CF, FG, GA, AH, HI, IB$. Furthermore, let the desired triangle be $\triangle XYZ$, with X closest to side BC, Y closest to side AC, and Z closest to side AB . Hence, the desired perimeter is

$XE + EF + FY + YG + GH + HZ + ZI + ID + DX = (DX + XE) + (FY + YG)$

since $HG = 55$, $EF = 15$, and $ID = 45$. Note that $\triangle AHG \sim \triangle BID \sim \triangle EFC \sim \triangle ABC$, so using similar triangle ratios, we find that $BI = HA = 30$, $BD = HG = 55$, $FC = \frac{45}{2}$, and $EC = \frac{55}{2}$.

We also notice that $\triangle EFC \sim \triangle YFG \sim \triangle EXD$ and $\triangle BID \sim \triangle HIZ$. Using similar triangles, we get that

$$FY + YG = \frac{GF}{FC} \cdot (EF + EC) = \frac{225}{45} \cdot \left(15 + \frac{55}{2}\right) = \frac{425}{2}$$

$$DX + XE = \frac{DE}{EC} \cdot (EF + FC) = \frac{275}{55} \cdot \left(15 + \frac{45}{2}\right) = \frac{375}{2}$$

$$HZ + ZI = \frac{IH}{BI} \cdot (ID + BD) = 2 \cdot (45 + 55) = 200$$

Hence, the desired perimeter is $200 + \frac{425 + 375}{2} + 115 = 600 + 115 = \boxed{715}$

8.We have $\frac{1 + \sqrt{3}i}{2} = \omega$ where $\omega = e^{\frac{i\pi}{3}}$ is a primitive 6th root of unity. Then we have

$$\begin{aligned} f(\omega) &= a\omega^{2018} + b\omega^{2017} + c\omega^{2016} \\ &= a\omega^2 + b\omega + c \end{aligned}$$

We wish to find $f(1) = a + b + c$. We first look at the real parts.

As $\text{Re}(\omega^2) = -\frac{1}{2}$ and $\text{Re}(\omega) = \frac{1}{2}$, we have $-\frac{1}{2}a + \frac{1}{2}b + c = 2015 \implies -a + b + 2c = 4030$. Looking at imaginary parts, we have $\text{Im}(\omega^2) = \text{Im}(\omega) = \frac{\sqrt{3}}{2}$, so $\frac{\sqrt{3}}{2}(a + b) = 2019\sqrt{3} \implies a + b = 4038$. As a and b do not exceed 2019, we must have $a = 2019$ and $b = 2019$.

Then $c=\dfrac{4030}{2}=2015$,

so $f(1)=4038+2015=6053 \implies f(1) \pmod{1000}=\boxed{053}$.

9.Every 20-pretty integer can be written in form $n=2^a5^bk$, where $a\geq 2$, $b\geq 1$, $\gcd(k,10)=1$, and $d(n)=20$, where $d(n)$ is the number of divisors of n. Thus, we have $20=(a+1)(b+1)d(k)$, using the fact that the divisor function is multiplicative. As $(a+1)(b+1)$ must be a divisor of 20, there are not many cases to check.If $a+1=4$, then $b+1=5$. But this leads to no solutions, as $(a,b)=(3,4)$ gives $2^35^4>2019$.

If $a+1=5$, then $b+1=2$ or 4. The first case gives $n=2^4\cdot 5^1\cdot p$ where p is a prime other than 2 or 5. Thus we have $80p<2019 \implies p=3,7,11,13,17,19,23$. The sum of all such n is $80(3+7+11+13+17+19+23)=5760$. In the second case $b+1=4$ and $d(k)=1$, and there is one solution $n=2^4\cdot 5^3=2000$.

If $a+1=10$, then $b+1=2$, but this gives $2^9\cdot 5^1>2019$. No other values for $a+1$ work.

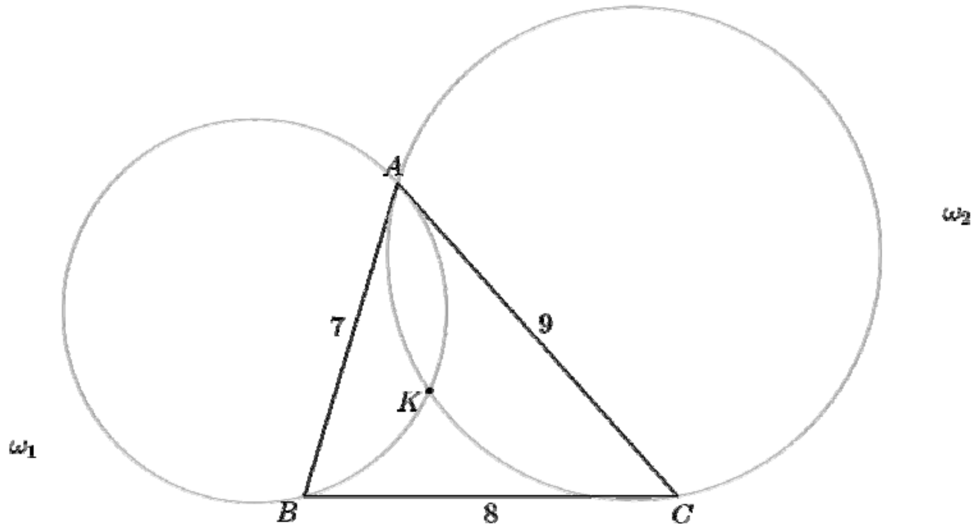
Then we have

$$\frac{S}{20}=\frac{80(3+7+11+13+17+19+23)+2000}{20}=372+100=\boxed{472}.$$

10.Note that if $\tan\theta$ is positive, then θ is in the first or third quadrant, so $0^\circ<\theta<90^\circ \pmod{180^\circ}$. Also notice that the only way $\tan(2^n\theta)$ can be positive for all n that are multiples of 3 is when $2^0\theta, 2^3\theta, 2^6\theta$, etc. are all the same value $\pmod{180^\circ}$. This happens if $8\theta=\theta \pmod{180^\circ}$, so $7\theta=0^\circ \pmod{180^\circ}$.

Therefore, the only possible values of theta between 0° and 90° are $\dfrac{180^\circ}{7}$, $\dfrac{360^\circ}{7}$, and $\dfrac{540^\circ}{7}$.

However $\frac{180^\circ}{7}$ does not work since $\tan 2\cdot\frac{180^\circ}{7}$ is positive, and $\frac{360^\circ}{7}$ does not work because $\tan 4\cdot\frac{360^\circ}{7}$ is positive. Thus, $\theta=\frac{540^\circ}{7}$. $540+7=\boxed{547}$.

11. 

-Diagram by Brendanb4321Note that from the tangency condition that the supplement of $\angle CAB$ with respects to lines AB and AC are equal to $\angle AKB$ and $\angle AKC$, respectively, so from tangent-chord, $\angle AKC=\angle AKB=180^{\circ}-\angle BAC$ Also note that $\angle ABK=\angle KAC$, so $\Delta AKB\sim\Delta CKA$. Using similarity ratios, we can easily find $AK^2=BK*KC$ However, since $AB=7$ and $CA=9$, we can use similarity ratios to get $BK=\frac{7}{9}AK, CK=\frac{9}{7}AK$ Now we use Law of Cosines on ΔAKB : From reverse Law of Cosines, $\cos\angle BAC=\frac{11}{21}\implies \cos\left(180^{\circ}-\angle BAC\right)=-\frac{11}{21}$. This gives us $AK^2+\frac{49}{81}AK^2+\frac{22}{27}AK^2=49\implies\frac{196}{81}AK^2=49$ $AK=\frac{9}{2}$ so our answer is $9+2=\boxed{011}$.

12.If the first term is x, then dividing through by x, we see that we can find the number of progressive sequences whose sum is $\frac{360}{x}-1$, and whose first term is not 1. If $a(k)$ denotes the number of progressive sequences whose sum is k and whose first term is not 1, then we can express the answer N as follows:

$$N = a(359) + a(179) + a(119) + a(89) + a(71) + a(59) + a(44) + a(39) + a(35) + a(29) + a(23) + a(19) + a(17) + a(14) + a(11) + a(9) + a(8) + a(7) + a(5) + a(4) + a(3) + a(2) + a(1) + 1$$

The $+1$ at the end accounts for the sequence whose only term is 360. Fortunately, many of these numbers are prime; we have $a(p) = 1$ for primes p as the only such sequence is "p" itself. Also, $a(1) = 0$. So we have

$$N = 15+a(119)+a(44)+a(39)+a(35)+a(14)+a(9)+a(8)+a(4)$$

For small k, $a(k)$ is easy to compute: $a(4) = 1$, $a(8) = 2$, $a(9) = 2$. For intermediate k (e.g. $k = 21$ below), $a(k)$ can be computed recursively using previously-computed values of $a(\cdot)$, similar to dynamic programming. Then we have

$$a(14) = 1 + a(6) = 1 + 2 = 3$$
$$a(35) = 1 + a(6) + a(4) = 1 + 2 + 1 = 4$$
$$a(39) = 2 + a(12) = 2 + 4 = 6$$
$$a(44) = 2 + a(21) + a(10) = 2 + 4 + 2 = 8$$
$$a(119) = 1 + a(16) + a(6) = 1 + 3 + 2 = 6$$

Thus the answer is $N = 15 + 6 + 8 + 6 + 4 + 3 + 2 + 2 + 1 = \boxed{047}$.

13.This problem is not difficult, but the calculation is tormenting.The actual size of the diagram doesn't matter. To make calculation easier, we discard the original area of the circle, 1, and assume the side length of the octagon is 2

Let r denotes the radius of the circle, O be the center of the circle.

$$r^2 = 1^2 + (\sqrt{2} + 1)^2 = 4 + 2\sqrt{2}$$

Now, we need to find the "D"shape, the small area enclosed by one side of the octagon and 1/8 of the circumference of the circle

$$D = \frac{1}{8}\pi r^2 - [A_1A_2O] = \frac{1}{8}\pi(4 + 2\sqrt{2}) - (\sqrt{2} + 1)$$

Let PU be the height of $\triangle A_1A_2P$, PV be the height of $\triangle A_3A_4P$, PW be the height of $\triangle A_6A_7P$,

From the 1/7 and 1/9 condition

we have

$$\triangle PA_1A_2 = \frac{\pi r^2}{7} - D = \frac{1}{7}\pi(4+2\sqrt{2}) - (\frac{1}{8}\pi(4+2\sqrt{2}) - (\sqrt{2}+1))$$

$$\triangle PA_3A_4 = \frac{\pi r^2}{9} - D = \frac{1}{9}\pi(4+2\sqrt{2}) - (\frac{1}{8}\pi(4+2\sqrt{2}) - (\sqrt{2}+1))$$

which gives

$$PU = (\frac{1}{7} - \frac{1}{8})\pi(4+2\sqrt{2}) + \sqrt{2} + 1$$

$$PV = (\frac{1}{9} - \frac{1}{8})\pi(4+2\sqrt{2}) + \sqrt{2} + 1$$

Now,
let A_1A_2 intersects A_3A_4 at X, A_1A_2 intersects A_6A_7 at Y, A_6A_7 intersects A_3A_4 at Z

Clearly, $\triangle XYZ$ is an isosceles right triangle, with right angle at X

and the height with regard to which shall be $3+2\sqrt{2}$

That $\frac{PU}{\sqrt{2}} + \frac{PV}{\sqrt{2}} + PW = 3 + 2\sqrt{2}$ is a common sense

which gives $PW = 3 + 2\sqrt{2} - \frac{PU}{\sqrt{2}} - \frac{PV}{\sqrt{2}}$

$$= 3 + 2\sqrt{2} - \frac{1}{\sqrt{2}}((\frac{1}{7} - \frac{1}{8})\pi(4+2\sqrt{2}) + \sqrt{2} + 1 + (\frac{1}{9} - \frac{1}{8})\pi(4+2\sqrt{2}) + \sqrt{2} + 1))$$

$$= 1 + \sqrt{2} - \frac{1}{\sqrt{2}}(\frac{1}{7} + \frac{1}{9} - \frac{1}{4})\pi(4+2\sqrt{2})$$

Now, we have the area for D and the area for $\triangle PA_6A_7$

we add them together

$$TargetArea = \frac{1}{8}\pi(4+2\sqrt{2}) - (\sqrt{2}+1) + (1+\sqrt{2}) - \frac{1}{\sqrt{2}}(\frac{1}{7}+\frac{1}{9}-\frac{1}{4})\pi(4+2\sqrt{2})$$

$$= (\frac{1}{8} - \frac{1}{\sqrt{2}}(\frac{1}{7}+\frac{1}{9}-\frac{1}{4}))TotalArea$$

The answer should therefore be $\frac{1}{8} - \frac{\sqrt{2}}{2}(\frac{16}{63} - \frac{16}{64}) = \frac{1}{8} - \frac{\sqrt{2}}{504}$

The final answer is, therefore, $\boxed{504}$

14. By the Chicken McNugget theorem, the least possible value of n such that 91 cents cannot be formed satisfies $5n - 5 - n = 91$, so $n = 24$. For values of n greater than 24, notice that if 91 cents cannot be formed, then any number $1 \bmod 5$ less than 91 also cannot be formed. The proof of this is that if any number $1 \bmod 5$ less than 91 can be formed, then we could keep adding 5 cent stamps until we reach 91 cents. However, since 91 cents is the greatest postage that cannot be formed, 96 cents is the first number that is $1 \bmod 5$ that can be formed, so it must be formed without any 5 cent stamps. There are few $(n, n+1)$ pairs, where $n \geq 24$, that can make 96 cents. These are cases where one of n and $n+1$ is a factor of 96, which are $(24,25),(31,32),(32,33),(47,48),(48,49),(95,96)$, and $(96,97)$. The last two obviously do not work since 92 through 94 cents also cannot be formed, and by a little testing, only $(24,25)$ and $(47,48)$ satisfy the condition that 91 cents is the greatest postage that cannot be formed, so $n = 24, 47$. $24 + 47 = \boxed{071}$.

15. $AP = a, AQ = b, cos\angle A = k$ Therefore $AB = \frac{b}{k}, AC = \frac{a}{k}$

By power of point, we have $AP * BP = XP * YP, AQ * CQ = YQ * XQ$ Which are simplified to

$$400 = \frac{ab}{k} - a^2$$

$525=\frac{ab}{k}-b^2$ Or $a^2=\frac{ab}{k}-400$ $b^2=\frac{ab}{k}-525$

(1)Or

$$k=\frac{ab}{a^2+400}=\frac{ab}{b^2+525}$$

Let $u=a^2+400=b^2+525$ Then,

$$a=\sqrt{u-400}, b=\sqrt{u-525}, k=\frac{\sqrt{(u-400)(u-525)}}{u}$$

In triangle APQ, by law of cosine

$$25^2=a^2+b^2-2abk$$

Pluging (1)

$$625=\frac{ab}{k}-400+\frac{ab}{k}-525-2abk$$

Or

$$\frac{ab}{k}-abk=775$$

Substitute everything by u

$$u-\frac{(u-400)(u-525)}{u}=775$$

The quadratic term is cancelled out after simplified

Which gives $u=1400$

Plug back in, $a=\sqrt{1000}, b=\sqrt{775}$

Then

$$AB*AC=\frac{a}{k}\frac{b}{k}=\frac{ab}{\frac{ab}{u}*\frac{ab}{u}}=\frac{u^2}{ab}=\frac{1400*1400}{\sqrt{1000*875}}=560\sqrt{14}$$

So the final answer is $560 + 14 = \boxed{574}$