2017 AMC 12A Problems and Solutions

Problem 1

Pablo buys popsicles for his friends. The store sells single popsicles for $1 each, 3-popsicle boxes for $2 each, and 5-popsicle boxes for $3. What is the greatest number of popsicles that Pablo can buy with $8?

（A）8 （B）11 （C）12 （D）13 （E）15

Problem 2

The sum of two nonzero real numbers is 4 times their product. What is the sum of the reciprocals of the two numbers?

（A）1 （B）2 （C）4 （D）8 （E）12

Problem 3

Ms. Carroll promised that anyone who got all the multiple choice questions right on the upcoming exam would receive an A on the exam. Which of these statements necessarily follows logically?

(A) If Lewis did not receive an A, then he got all of the multiple choice questions wrong.
(B) If Lewis did not receive an A, then he got at least one of the multiple choice questions wrong.
(C) If Lewis got at least one of the multiple choice questions wrong, then he did not receive an A.
(D) If Lewis received an A, then he got all of the multiple choice questions right.
(E) If Lewis received an A, then he got at least one of the multiple choice questions right.

Problem 4

Jerry and Silvia wanted to go from the southwest corner of a square field to the northeast corner. Jerry walked due east and then due north to reach the goal, but Silvia headed northeast and reached the goal walking in a straight line. Which of the following is closest to how much shorter Silvia's trip was, compared to Jerry's trip?

（A）30% （B）40% （C）50% （D）60% （E）70%

Problem 5

At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur?

（A）240 （B）245 （C）250 （D）480 （E）490

Problem 6

Joy has 30 thin rods, one each of every integer length from 1 cm through 30 cm. She places the rods with lengths 3 cm, 7 cm, and 15 cm on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod?

(A) 16 (B) 17 (C) 18 (D) 19 (E) 20

Problem 7

Define a function on the positive integers recursively by $f(1)=2$, $f(n)=f(n-1)+2$ if n is even, and $f(n)=f(n-2)+2$ if n is odd and greater than 1. What is $f(2017)$?

(A) 2017 (B) 2018 (C) 4034 (D) 4035 (E) 4036

Problem 8

The region consisting of all points in three-dimensional space within 3 units of line segment $\overline{AB}$ has volume 216π. What is the length AB?

(A) 6 (B) 12 (C) 18 (D) 20 (E) 24

Problem 9

Let S be a set of points (x,y) in the coordinate plane such that two of the three quantities $3,\ x+2,$ and $y-4$ are equal and the third of the three quantities is no greater than this common value. Which of the following is a correct description for S?

(A) a single point **(B)** two intersecting lines

(C) three lines whose pairwise intersections are three distinct points

(D) a triangle **(E)** three rays with a common endpoint

Problem 10

Chloé chooses a real number uniformly at random from the interval $[0,2017]$. Independently, Laurent chooses a real number uniformly at random from the interval $[0,4034]$. What is the probability that Laurent's number is greater than Chloé's number?

(A) $\frac{1}{2}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{5}{6}$ **(E)** $\frac{7}{8}$

Problem 11

Claire adds the degree measures of the interior angles of a convex polygon and arrives at a sum of 2017. She then discovers that she forgot to include one angle. What is the degree measure of the forgotten angle?

(A) 37 (B) 63 (C) 117 (D) 143 (E) 163

Problem 12

There are 10 horses, named Horse 1, Horse 2, . . ., Horse 10. They get their names from how many minutes it takes them to run one lap around a circular race track: Horse k runs one lap in exactly k minutes. At time 0 all the horses are together at the starting point on the track. The horses start running in the same direction, and they keep running around the circular track at their constant speeds. The least time $S > 0$, in minutes, at which all 10 horses will again simultaneously be at the starting point is $S = 2520$. Let $T > 0$ be the least time, in minutes, such that at least 5 of the horses are again at the starting point. What is the sum of the digits of T?

（A）2　　（B）3　　（C）4　　（D）5　　（E）6

Problem 13

Driving at a constant speed, Sharon usually takes 180 minutes to drive from her house to her mother's house. One day Sharon begins the drive at her usual speed, but after driving 1/3 of the way, she hits a bad snowstorm and reduces her speed by 20 miles per hour. This time the trip takes her a total of 278 minutes. How many miles is the drive from Sharon's house to her mother's house?

（A）132　　（B）135　　（C）138　　（D）141　　（E）144

Problem 14

Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. How many ways are there for the five of them to sit in a row of 5 chairs under these conditions?

（A）12　　（B）16　　（C）28　　（D）32　　（E）40

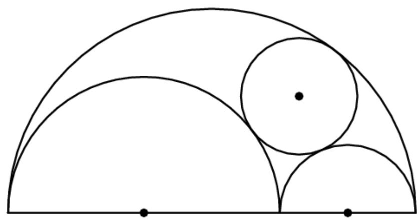
Problem 15

Let $f(x) = \sin x + 2\cos x + 3\tan x$, using radian measure for the variable x. In what interval does the smallest positive value of x for which $f(x) = 0$ lie?

（A）(0,1)　　（B）(1,2)　　（C）(2,3)　　（D）(3,4)　　（E）(4,5)

Problem 16

In the figure below, semicircles with centers at A and B and with radii 2 and 1, respectively, are drawn in the interior of, and sharing bases with, a semicircle with diameter JK. The two smaller semicircles are externally tangent to each other and internally tangent to the largest semicircle. A circle centered at P is drawn externally tangent to the two smaller semicircles and internally tangent to the largest semicircle. What is the radius of the circle centered at P?



(A) $\frac{3}{4}$　　**(B)** $\frac{6}{7}$　　**(C)** $\frac{1}{2}\sqrt{3}$　　**(D)** $\frac{5}{8}\sqrt{2}$　　**(E)** $\frac{11}{12}$

Problem 17

There are 24 different complex numbers z such that $z^{24} = 1$. For how many of these is z^6 a real number?

（A）0　（B）4　（C）6　（D）12　（E）24

Problem 18

Let S(n) equal the sum of the digits of positive integer n. For example, S(1507)=13. For a particular positive integer n, S(n)=1274. Which of the following could be the value of $S(n+1)$?

（A）1　（B）3　（C）12　（D）1239　（E）1265

Problem 19

A square with side length x is inscribed in a right triangle with sides of length 3,4, and 5 so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length 3,4, and 5 so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$?

$\textbf{(A)}\ \frac{12}{13} \qquad \textbf{(B)}\ \frac{35}{37} \qquad \textbf{(C)}\ 1 \qquad \textbf{(D)}\ \frac{37}{35} \qquad \textbf{(E)}\ \frac{13}{12}$

Problem 20

How many ordered pairs (a,b) such that a is a positive real number and b is an integer between 2 and 200, inclusive, satisfy the equation $(\log_b a)^{2017}=\log_b(a^{2017})$?

（A）198　（B）199　（C）398　（D）399　（E）597

Problem 21

A set S is constructed as follows. To begin, $S=\{0,10\}$. Repeatedly, as long as possible, if x is an integer root of some polynomial $a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$ for some $n\geq 1$, all of whose coefficients a_i are elements of S, then x is put into S. When no more elements can be added to S, how many elements does S have?

（A）4　（B）5　（C）7　（D）9　（E）11

Problem 22

A square is drawn in the Cartesian coordinate plane with vertices at (2,2),(-2,2),(-2,-2),(2,-2). A particle starts at (0,0). Every second it moves with equal probability to one of the eight lattice points (points with integer coordinates) closest to its current position, independently of its previous moves. In other words, the probability is 1/8 that the particle will move from (x,y) to each of

$(x, y+1)$, $(x+1, y+1)$, $(x+1, y)$, $(x+1, y-1)$, $(x, y-1)$, $(x-1, y-1)$, $(x-1, y)$, or $(x-1, y+1)$. The particle will eventually hit the square for the first time, either at one of the 4 corners of the square or at one of the 12 lattice points in the interior of one of the sides of the square. The probability that it will hit at a corner rather than at an interior point of a side is m/n, where m and n are relatively prime positive integers. What is $m+n$?

（A）4　（B）5　（C）7　（D）15　（E）39

Problem 23

For certain real numbers a, b, and c, the polynomial $g(x) = x^3 + ax^2 + x + 10$ has three distinct roots, and each root of $g(x)$ is also a root of the polynomial $f(x) = x^4 + x^3 + bx^2 + 100x + c$. What is $f(1)$?

（A）-9009　（B）-8008　（C）-7007　（D）-6006　（E）-5005

Problem 24

Quadrilateral *ABCD* is inscribed in circle *O* and has side lengths *AB*=3，*BC*=2，*CD*=6, and *DA* =*8*. Let X and Y be points on $\overline{BD}$ such that *DX/BD*=1/4 and *BY/BD*=11/36. Let E be the intersection of line AX and the line through Y parallel to $\overline{AD}$. Let F be the intersection of line CX and the line through E parallel to $\overline{AC}$. Let G be the point on circle O other than C that lies on line CX. What is $XF \cdot XG$?

$\textbf{(A)}\ 17 \qquad \textbf{(B)}\ \frac{59-5\sqrt{2}}{3} \qquad \textbf{(C)}\ \frac{91-12\sqrt{3}}{4} \qquad \textbf{(D)}\ \frac{67-10\sqrt{2}}{3} \qquad \textbf{(E)}\ 18$

Problem 25

The vertices V of a centrally symmetric hexagon in the complex plane are given by $V=\left\{\sqrt{2}i,-\sqrt{2}i,\frac{1}{\sqrt{8}}(1+i),\frac{1}{\sqrt{8}}(-1+i),\frac{1}{\sqrt{8}}(1-i),\frac{1}{\sqrt{8}}(-1-i)\right\}.$

For each j, $1 \le j \le 12$, an element z_j is chosen from V at random, independently of the other choices. Let $P=\prod_{j=1}^{12} z_j$ be the product of the 12 numbers selected. What is the probability that $P=-1$?

$\textbf{(A)}\ \frac{5 \cdot 11}{3^{10}} \qquad \textbf{(B)}\ \frac{5^2 \cdot 11}{2 \cdot 3^{10}} \qquad \textbf{(C)}\ \frac{5 \cdot 11}{3^{9}} \qquad \textbf{(D)}\ \frac{5 \cdot 7 \cdot 11}{2 \cdot 3^{10}} \qquad \textbf{(E)}\ \frac{2^2 \cdot 5 \cdot 11}{3^{10}}$

2017 AMC 12A Solutions

Problem 1

$3 boxes give us the most popsicles/dollar, so we want to buy as many of those as possible. After buying 2, we have $2 left. We cannot buy a third $3 box, so we opt for the $2 box instead (since it has a higher popsicles/dollar ratio than the $1 pack). We're now out of money. We bought 5+5+3=13 popsicles, so the answer is D13.

Problem 2

Solution1

Let the two real numbers be x, y. We are given that $x + y = 4xy$, and dividing both sides by xy, $\frac{x}{xy} + \frac{y}{xy} = 4.$

$\frac{1}{y} + \frac{1}{x} = \boxed{\textbf{(C) } 4}.$

Note: we can easily verify that this is the correct answer; for example, 1/2 and 1/2 work, and the sum of their reciprocals is 4.

Solution 2

Instead of using algebra, another approach at this problem would be to notice the fact that one of the nonzero numbers has to be a fraction. See for yourself. And by looking into fractions, we immediately see that $\frac{1}{3}$ and 1would fit the rule. C14

Solution 3

Notice that from the information given above, $x + y = 4xy$

Because the sum of the reciprocals of two numbers is just the sum of the two numbers over the product of the two numbers or $\frac{x+y}{xy}$

We can solve this by substituting $x + y \implies 4xy.$

Our answer is simply $\frac{4xy}{xy} \implies 4.$

Therefore, the answer is $\boxed{\textbf{(C) } 4}.$

Problem 3

Rewriting the given statement: "if someone got all the multiple choice questions right on the upcoming exam then he or she would receive an A on the exam." If that someone is Lewis the statement becomes: "if Lewis got all the multiple choice questions right, then he got an A on the exam." The contrapositive: "If Lewis did not receive an A, then he got at least one of the multiple choice questions wrong (did not get all of them right)" must also be true leaving B as the correct answer. B is also equivalent to the contrapositive of the original statement, which implies that it must be true, so the answer is B

Problem 4

Let j represent how far Jerry walked, and s represent how far Sylvia walked. Since the field is a square, and Jerry walked two sides of it, while Silvia walked the diagonal, we can simply define the side of the square field to be one, and find the distances they walked. Since Jerry walked two sides, $j = 2$ Since Silvia walked the diagonal, she walked the hypotenuse of a 45, 45, 90 triangle with leg length 1. Thus, $s = \sqrt{2} = 1.414...$ We can then take $\frac{j - s}{j} \approx \frac{2 - 1.4}{2} = 0.3 \implies \boxed{\textbf{(A) } 30\%}$

Problem 5

Solution 1

Each one of the ten people has to shake hands with all the 20 other people they don't know. So $10 \cdot 20 = 200$. From there, we calculate how many handshakes occurred between the people who don't know each other. This is simply counting how many ways to choose two people to shake hands, or $\binom{10}{2} = 45$. Thus the answer is $200 + 45 = \boxed{\textbf{(B) } 245}$.

Solution 2

We can also use complementary counting. First of all, $\binom{30}{2} = 435$ handshakes or hugs occur. Then, if we can find the number of hugs, then we can subtract it from 435 to find the handshakes. Hugs only happen between the 20 people who know each other, so there are $\binom{20}{2} = 190$ hugs. $435 - 190 = \boxed{\textbf{(B) } 245}$.

Solution 3

We can focus on how many handshakes the 10 people get.
The 1st person gets 29 handshakes.
2nd gets 28
......
And the 10th receives 20 handshakes.
We can write this as the sum of an arithmetic sequence.
$\frac{10(20+29)}{2} \implies 5(49) \implies 245.$ Therefore, the answer is $\boxed{\textbf{(B) } 245}$

Problem 6

The triangle inequality generalizes to all polygons, so $x < 3 + 7 + 15$ and $x + 3 + 7 > 15$ to get $5 < x < 25$. Now, we know that there are 19 numbers between 5 and 25 exclusive, but we must subtract 2 to account for the 2 lengths already used that are between those numbers, which gives 19-2=17 B

Problem 7

This is a recursive function, which means the function is used to evaluate itself. To solve this, we must identify the base case, $f(1) = 2$. We also know that when n is odd, $f(n) = f(n-2) + 2$. Thus we know that $f(2017) = f(2015) + 2$. Thus we know that n will always be odd in the recursion of $f(2017)$, and we add 2 each recursive cycle, which there are 1008 of. Thus the answer is $1008 * 2 + 2 = 2018$, which is answer $\boxed{\textbf{(B)}}$.

Problem 8

In order to solve this problem, we must first visualize what the region contained looks like. We know that, in a three dimensional plane, the region consisting of all points within 3 units of a point would be a sphere with radius 3. However, we need to find the region containing all points within 3 units of a segment. It can be seen that our region is a cylinder with two hemispheres on either end. We know the volume of our region, so we set up the following equation (the volume of our cylinder + the volume of our two hemispheres will equal 216π):

$\frac{4\pi}{3} \cdot 3^3 + 9\pi x = 216\pi$, where x is equal to the length of our line segment.

Solving, we find that $x = \boxed{\textbf{(D)}\ 20}$.

Problem 9

If the two equal values are 3 and $x+2$, then $x=1$. Also, $y-4 \le 3$ because 3 is the common value. Solving for y, we get $y \le 7$. Therefore the portion of the line $x=1$ where $y \le 7$ is part of S. This is a ray with an endpoint of $(1,7)$.

Similar to the process above, we assume that the two equal values are 3 and $y-4$. Solving the equation $3 = y-4$ then $y=7$. Also, $x+2 \le 3$ because 3 is the common value. Solving for x, we get $x \le 1$. Therefore the portion of the line $y=7$ where $x \le 1$ is also part of S. This is another ray with the same endpoint as the above ray: $(1,7)$.

If $x+2$ and $y-4$ are the two equal values, then $x+2 = y-4$. Solving the equation for y, we get $y = x+6$. Also $3 \le y-4$ because $y-4$ is one way to express the common value. Solving for y, we get $y \ge 7$. Therefore the portion of the line $y = x+6$ where $y \ge 7$ is part of S like the other two rays. The lowest possible value that can be achieved is also $(1,7)$.

Since S is made up of three rays with common endpoint $(1,7)$, the answer is $\boxed{\textbf{(E)}\ \text{three rays with a common endpoint}}$

Problem 10

Solution 1

Denote "winning" to mean "picking a greater number". There is a $\frac{1}{2}$ chance that Laurent chooses a number in the interval $[2017, 4034]$. In this case, Chloé cannot possibly win, since the maximum number she can pick is 2017. Otherwise, if Laurent picks a number in the interval $[0, 2017]$, with probability $\frac{1}{2}$, then the two people are symmetric, and each has a $\frac{1}{2}$ chance of winning. Then, the total probability is $\frac{1}{2} * 1 + \frac{1}{2} * \frac{1}{2} = \boxed{\textbf{(C)}\ \frac{3}{4}}$

Solution 2

We can use geometric probability to solve this. Suppose a point (x, y) lies in the $xy-$plane. Let x be Chloe's number and y be Laurent's number. Then obviously we want $y > x$, which basically gives us a region above a line. We know that Chloe's number is in the interval $[0, 2017]$ and Laurent's number is in the interval $[0, 4034]$, so we can create a rectangle in the plane, whose length is 2017 and whose width is 4034. Drawing it out, we see that it is easier to find the probability that Chloe's number is greater than Laurent's number and subtract this probability from 1. The probability that Chloe's number is larger than Laurent's number is simply the area of the region under the line $y > x$, which is $\frac{2017 \cdot 2017}{2}$. Instead of bashing this out we know that the rectangle has area $2017 \cdot 4034$. So the probability that Laurent has a smaller number is $\frac{2017 \cdot 2017}{2 \cdot 2017 \cdot 4034}$. Simplifying the expression yields $\frac{1}{4}$ and so $1 - \frac{1}{4} = \boxed{\textbf{(C)}\ \frac{3}{4}}$.

Solution3

Scale down by 2017 to get that Chloe picks from $[0, 1]$ and Laurent picks from $[0, 2]$. There are an infinite number of cases for the number that Chloe picks, but they are all centered around the average of 0.5. Therefore, Laurent has a range of 0.5 to 2 to pick from, on average, which is a length of $2 - 0.5 = 1.5$ out of a total length of $2 - 0 = 2$. Therefore, the probability is 1.5/2=3/4 C

Problem 11

We know that the sum of the interior angles of the polygon is a multiple of 180. Note that $\left\lceil \frac{2017}{180} \right\rceil = 12$ and $180 \cdot 12 = 2160$, so the angle Claire forgot is $\equiv 2160 - 2017 = 143 \mod 180$. Since the polygon is convex, the angle is ≤ 180, so the answer is $\boxed{(D)\ =\ 143}$.

Problem 12

Solution 1

If we have horses, $a_1, a_2, \ldots, a_n$, then any number that is a multiple of the all those numbers is a time when all horses will meet at the starting point. The least of these numbers is the LCM. To minimize the LCM, we need the smallest primes, and we need to repeat them a lot. By inspection, we find that $\text{LCM}(1, 2, 3, 2 \cdot 2, 2 \cdot 3) = 12$.

Finally, 1+2=3 B.

Solution 2

We are trying to find the smallest number that has 5 one-digit divisors. Therefore we try to find the LCM for smaller digits, such as $1, 2$, 3, or 4. We quickly consider 12 since it is the smallest number that is the LCM of 1, 2, 3 and 4. Since 12 has 5 single-digit divisors, namely 1, 2, 3, 4, and 6, our answer is 1+2=3 B.

Problem 13

Let total distance be x. Her speed in miles per minute is $\frac{x}{180}$. Then, the distance that she drove before hitting the snowstorm is $\frac{x}{3}$. Her speed in snowstorm is reduced 20 miles per hour, or $\frac{1}{3}$ miles per minute. Knowing it took her 276 minutes in total, we create equation:

$$\text{Time before Storm} + \text{Time after Storm} = \text{Total Time} \Longrightarrow$$

$$\frac{\text{Distance before Storm}}{\text{Speed before Storm}} + \frac{\text{Distance in Storm}}{\text{Speed in Storm}} = \text{Total Time} \Longrightarrow \frac{\frac{x}{3}}{\frac{x}{180}} + \frac{\frac{2x}{3}}{\frac{x}{180} - \frac{1}{3}} = 276$$

Solving equation, we get $x = 135 \Longrightarrow \boxed{B}$.

Problem 14

Solution 1

For notation purposes, let Alice be A, Bob be B, Carla be C, Derek be D, and Eric be E. We can split this problem up into two cases:

$\textbf{Case 1:}$ A sits on an edge seat.

Then, since B and C can't sit next to A, that must mean either D or E sits next to A. After we pick either D or E, then either B or C must sit next to D/E. Then, we can arrange the two remaining people in two ways. Since there are two different edge seats that A can sit in, there are a total of $2 \cdot 2 \cdot 2 \cdot 2 = 16$.

$\textbf{Case 2:}$ A does not sit in an edge seat.

In this case, then only two people that can sit next to A are D and E, and there are two ways to permute them, and this also handles the restriction that D can't sit next to E. Then, there are two ways to arrange B and C, the remaining people. However, there are three initial seats that A can sit in, so there are $3 \cdot 2 \cdot 2 = 12$ seatings in this case.

Adding up all the cases, we have $16 + 12 = \boxed{\textbf{(C) } 28}$.

Solution 2

Label the seats 1 through 5. The number of ways to seat Derek and Eric in the five seats with no restrictions is $5 * 4 = 20$. The number of ways to seat Derek and Eric such that they sit next to each other is 8 (which can be figure out quickly), so the number of ways such that Derek and Eric don't sit next to each other is $20 - 8 = 12$. Note that once Derek and Eric are seated, there are three cases.

The first case is that they sit at each end. There are two ways to seat Derek and Eric. But this is impossible because then Alice, Bob, and Carla would have to sit in some order in the middle three seats which would lead to Alice sitting next to Bob or Carla, a contradiction. So this case gives us 0 ways.

Another possible case is if Derek and Eric seat in seats 2 and 4 in some order. There are 2 possible ways to seat Derek and Eric like this. This leaves Alice, Bob, and Carla to sit in any order in the remaining three seats. Since no two of these three seats are consecutive, there are $3! = 6$ ways to do this. So the second case gives us $2 * 6 = 12$ total ways for the second case.

The last case is if once Derek and Eric are seated, exactly one pair of consecutive seats is available. There are $12 - 2 - 2 = 8$ ways to seat Derek and Eric like this. Once they are seated like this, Alice cannot not sit in one of the two consecutive available seats without sitting next to Bob and Carla. So Alice has to sit in the other remaining chair. Then, there are two ways to seat Bob and Carla in the remaining two seats (which are consecutive). So this case gives us 8*2=16 ways.

So in total there are 12+16=28. So our answer is $\boxed{\textbf{(C) } 28}$.

Problem 15

We must first get an idea of what $f(x)$ looks like:

Between 0 and 1, $f(x)$ starts at 2 and increases; clearly there is no zero here.

Between 1 and $\frac{\pi}{2}$, $f(x)$ starts at a positive number and increases to ∞; there is no zero here either.

Between $\frac{\pi}{2}$ and 3, $f(x)$ starts at $-\infty$ and increases to some negative number; there is no zero here either.

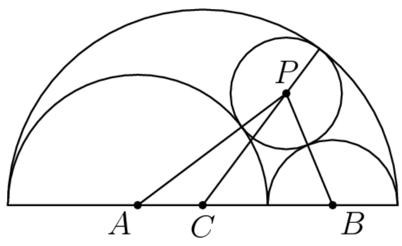
Between 3 and π, $f(x)$ starts at some negative number and increases to -2; there is no zero here either.

Between π and $\pi + \frac{\pi}{4} < 4$, $f(x)$ starts at -2 and increases to $-\frac{\sqrt{2}}{2} + 2\left(-\frac{\sqrt{2}}{2}\right) + 3(1) = 3\left(1 - \frac{\sqrt{2}}{2}\right) > 0$. There is a zero here by the Intermediate Value Theorem. Therefore, the answer is $\boxed{(D)}$.

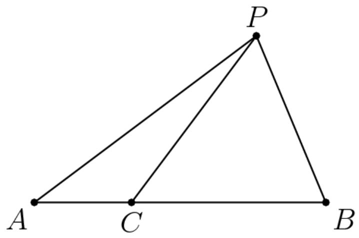
Problem 16

Solution 1

Connect the centers of the tangent circles! (call the center of the large circle C)



Notice that we don't even need the circles anymore; thus, draw triangle ΔABP with cevian PC:



and use Stewart's Theorem:

$$AB \cdot AC \cdot BC + AB \cdot CP^2 = AC \cdot BP^2 + BC \cdot AP^2$$

From what we learned from the tangent circles, we have $AB = 3,\ \ AC = 1,\ \ BC = 2,$ $AP = 2 + r,\ \ BP = 1 + r,$ and $CP = 3 - r,$ where r is the radius of the circle centered at P that we seek.

Thus:

$$3 \cdot 1 \cdot 2 + 3(3 - r)^2 = 1(1 + r)^2 + 2(2 + r)^2$$

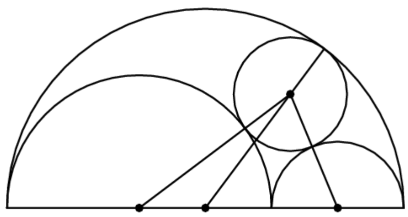
$$6 + 3\left(9 - 6r + r^2\right) = \left(1 + 2r + r^2\right) + 2\left(4 + 4r + r^2\right)$$

$$33 - 18r + 3r^2 = 9 + 10r + 3r^2$$

$$28r = 24$$

$$r = \boxed{\frac{6}{7}} \rightarrow \boxed{\textbf{(B)}}$$

Solution 2



Like the solution above, connecting the centers of the circles results in triangle ΔABP with cevian PC. The two triangles ΔAPC and ΔABP share angle A, which means we can use Law of Cosines to set up a system of 2 equations that solve for r respectively:

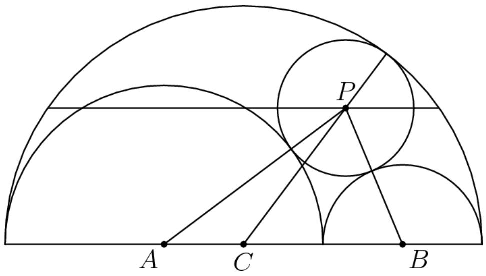
$(2 + r)^2 + 1^2 - 2(2 + r)(1)cosA = (3 - r)^2$ (notice that the diameter of the largest semicircle is 6, so its radius is 3 and PC is 3 − r)

$(2 + r)^2 + 3^2 - 2(2 + r)(3)cosA = (r + 1)^2$

We can eliminate the extra variable of angle A by multiplying the first equation by 3 and subtracting the second from it. Then, expand to find r:

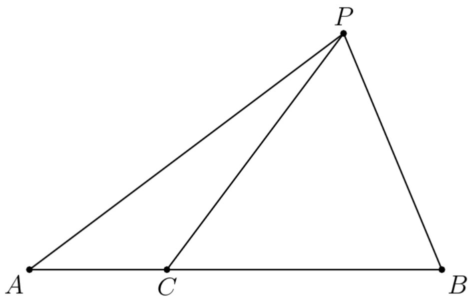
$2(r^2 + 4r + 4) - 6 = 2r^2 - 20r + 26\ \ 8r + 2 = -20r + 26\ \ 28r = 24,$ so $r = 6/7$ $\boxed{(B)}$

Solution 3



Let C be the center of the largest semicircle and r be the radius of $\circ P$. We know that $AC = 1$, $CB = 2$, $AP = r + 2$, $BP = r + 1$, and $CP = 3 - r$. Notice that ΔACP and ΔCBP are bounded by the same two parallel lines, so these triangles have the same heights. Because the bases of these two triangles (that have the same heights) differ by a factor of 2, the area of ΔCBP must be twice that of ΔACP, since the area of a triangle is $\frac{1}{2} Base \cdot Height$.

Again, we don't need to look at the circle and the semicircles anymore; just focus on the triangles.



Let A_1 equal to the area of ΔACP and A_2 equal to the area of ΔCBP. Heron's Formula states that the area of an triangle with sides a b and c is

$$\sqrt{s(s-a)(s-b)(s-c)}$$

where s, or the semiperimeter, is $\dfrac{a+b+c}{2}$

The semiperimeter s_1 of ΔACP is

$$\frac{[(r+2)+(3-r)+1]}{2} = \frac{6}{2} = 3$$

Use Heron's Formula to obtain

$$A_1 = \sqrt{3(2)(3-2-r)(3-3+r)} = \sqrt{6r(1-r)} = \sqrt{6r-6r^2}$$

Using Heron's Formula again, find the area of ΔCBP with sides $r+1$, 2, and $3-r$.

$$s_2 = \frac{(r+1)+2+(3-r)}{2} = 3$$

$$A_2 = \sqrt{3(3-2)(3-1-r)(3-3+r)} = \sqrt{3(2r-r^2)} = \sqrt{6r-3r^2}$$

Now,

$$2 \cdot A_1 = A_2$$

$$2\sqrt{6r-6r^2} = \sqrt{6r-3r^2}$$

$$4(6r-6r^2) = 6r-3r^2$$

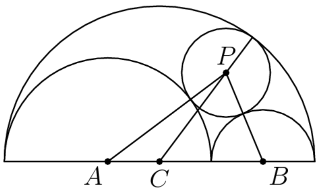
$$24r-24r^2 = 6r-3r^2$$

$$18r = 21r^2$$

$$r = \frac{18}{21} = \boxed{\frac{6}{7}} \rightarrow \boxed{\textbf{(B)}}$$

Solution 4

Let C, the center of the large semicircle, to be at $(0,0)$, and P to be at (h,k).



Therefore A is at $(-1,0)$ and B is at $(2,0)$.

Let the radius of circle P be r.

Using Distance Formula, we get the following system of three equations:

$$h^2+k^2 = (3-r)^2, (h+1)^2+k^2 = (r+2)^2, (h-2)^2+k^2 = (r+1)^2$$

By simplifying, we get

$$h^2+k^2 = r^2-6r+9, h^2+2r+1+k^2 = r^2+4r+4, h^2-4r+4+k^2 = r^2+2r+1$$

By subtracting the first equation from the second and third equations, we get

$$8r = -4h+12, 10r = 2h+6$$

which simplifies to

$$2r = 3-h, 5r = h+3$$

When we add these two equations, we get

$$7r = 6$$

Problem 17

Solution 1

Note that these z such that $z^{24}=1$ are $e^{\frac{n i \pi}{12}}$ for integer $0 \leq n < 24$. So

$z^6 = e^{\frac{n i \pi}{2}}$

This is real if $\dfrac{n}{2} \in \mathbb{Z} \Leftrightarrow (n$ is even$)$. Thus, the answer is the number of even $0 \leq n < 24$

which is $\boxed{(D) = \ 12}$.

Solution 2

$z = \sqrt[24]{1} = 1^{\frac{1}{24}}$

By Euler's identity, $1 = e^{0\times i} = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi)$, where k is an integer.

Using De Moivre's Theorem, we have $z = 1^{\frac{1}{24}} = \cos(\frac{k\pi}{12}) + i\sin(\frac{k\pi}{12})$, where $0 \leq k < 24$ that produce 24 unique results.

Using De Moivre's Theorem again, we have $z^6 = \cos(\frac{k\pi}{2}) + i\sin(\frac{k\pi}{2})$

For z^6 to be real, $\sin(\frac{k\pi}{2})$ has to equal 0 to negate the imaginary component. This occurs whenever $\frac{k\pi}{2}$ is an integer multiple of π, requiring that k is even. There are exactly $\boxed{12}$ even values of k on the interval $0 \leq k < 24$, so the answer is $\boxed{(D)}$.

Solution 3

From the start, recall from the Fundamental Theorem of Algebra that $z^{24} = 1$ must have 24 solutions (and these must be distinct since the equation factors into $0 = (z-1)(z^{23} + z^{22} + z^{21}... + z + 1))$, or notice that the question is simply referring to the 24th roots of unity, of which we know there must be 24. Notice that $1 = z^{24} = (z^6)^4$, so for any solution z, z^6 will be one of the 4th roots of unity $(1, i, -1,$ or $-i)$. Then 6 solutions z will satisfy $z^6 = 1$, 6 will satisfy $z^6 = -1$ (and this is further justified by knowledge of the 6th roots of unity), so there must be $\boxed{(D)\ 12}$ such z.

Problem 18

Solution 1

Note that $n \equiv S(n) \pmod 9$. This can be seen from the fact that $\sum_{k=0}^{n} 10^k a_k \equiv \sum_{k=0}^{n} a_k \pmod 9$. Thus, if $S(n) = 1274$, then $n \equiv 5 \pmod 9$, and thus $n + 1 \equiv S(n+1) \equiv 6 \pmod 9$. The only answer choice that is $6 \pmod 9$ is $\boxed{\textbf{(D) } 1239}$.

Solution 2

One divisibility rule for division that we can use in the problem is that a multiple of 9 has its digit always add up to a multiple of 9. We can find out that the least number of digits the number N is 142, with 141 9's and 1 5, assuming the rule above. No matter what arrangement or different digits we use, the divisor rule stays the same. To make the problem simpler, we can just use the 141 9's and 1 5. By randomly mixing the digits up, we are likely to get: $9999...9995999...9999$. By adding 1 to this number, we get: $9999...9996000...0000$. Knowing that this number is ONLY divisible by 9 when 6 is subtracted, we can subtract 6 from every available choice, and see if the number is divisible by 9 afterwards. After subtracting 6 from every number, we can conclude that 1233 (originally 1239) is the only number divisible by 9. So our answer is $\boxed{\textbf{(D) } 1239}$.

Solution 3

The number n can be viewed as having some unique digits in the front, following by a certain number of nines. We can then evaluate each potential answer choice.

If $\boxed{\textbf{(A) } 1}$ is correct, then n must be some number $99999999...9$, because when we add one to $99999999...9$ we get $10000000...00$. Thus, if 1 is the correct answer, then the equation $9x = 1274$ must have an integer solution (i.e. 1274 must be divisible by 9). But since it does not, 1 is not the correct answer.

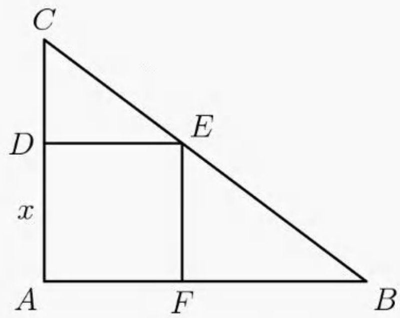
If $\boxed{\textbf{(B) } 3}$ is correct, then n must be some number $29999999...9$, because when we add one to $29999999...9$, we get $30000000...00$. Thus, if 2 is the correct answer, then the equation $2 + 9x = 1274$ must have an integer solution. But since it does not, 3 is not the correct answer.

Based on what we have done for evaluating the previous two answer choices, we can create an equation we can use to evaluate the final three possibilities. Notice that if $S(n+1) = N$, then n must be a number whose initial digits sum to $N - 1$, and whose other, terminating digits, are all 9. Thus, we can evaluate the three final possibilities by seeing if the equation $(N-1) + 9x = 1274$ has an integer solution.

The equation does not have an integer solution for $N = 12$, so $\boxed{\textbf{(C) } 12}$ is not correct. However, the equation does have an integer solution for $N = 1239$ $(x = 4)$, so $\boxed{\textbf{(D) } 1239}$ is the answer.

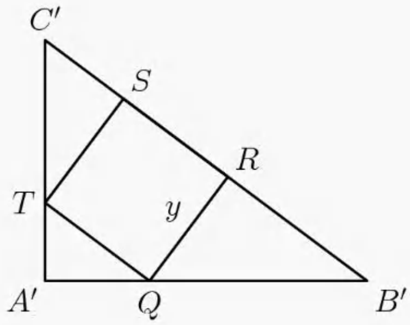
Problem 19

Analyze the first right triangle.



Note that $\triangle ABC$ and $\triangle FBE$ are similar, so $\frac{BF}{FE} = \frac{AB}{AC}$. This can be written as $\frac{4-x}{x} = \frac{4}{3}$. Solving, $x = \frac{12}{7}$.

Now we analyze the second triangle.



Similarly, $\triangle A'B'C'$ and $\triangle RB'Q$ are similar, so $RB' = \frac{4}{3}y$, and $C'S = \frac{3}{4}y$. Thus, $C'B' = C'S + SR + RB' = \frac{4}{3}y + y + \frac{3}{4}y = 5$. Solving for y, we get $y = \frac{60}{37}$. Thus, $\frac{x}{y} = \boxed{\textbf{(D)}\ \frac{37}{35}}$.

Problem 20

By the properties of logarithms, we can rearrange the equation to read $x^{2017} = 2017x$ with $x = \log_b a$. If $x \neq 0$, we may divide by it and get $x^{2016} = 2017$, which implies $x = \pm \sqrt[2016]{2017}$. Hence, we have 3 possible values x, namely

$$x = 0, \qquad x = 2017^{\frac{1}{2016}}, \text{ and } \quad x = -2017^{\frac{1}{2016}}.$$

Since $\log_b a = x$ is equivalent to $a = b^x$, each possible value x yields exactly 199 solutions (b, a), as we can assign $a = b^x$ to each $b = 2, 3, \ldots, 100$. In total, we have $3 \cdot 199 = \boxed{\textbf{(E)}\ 597}$ solutions.

Problem 21

At first, $S = \{0, 10\}$.

$10x + 10$	has root	$x = -1$	so now	$S = \{-1, 0, 10\}$
$-x^{10} - x^9 - x^8 - x^7 - x^6 - x^5 - x^4 - x^3 - x^2 - x + 10$	has root	$x = 1$	so now	$S = \{-1, 0, 1, 10\}$
$x + 10$	has root	$x = -10$	so now	$S = \{-10, -1, 0, 1, 10\}$
$x^4 - x^2 - x - 10$	has root	$x = 2$	so now	$S = \{-10, -1, 0, 1, 2, 10\}$
$x^4 - x^2 + x - 10$	has root	$x = -2$	so now	$S = \{-10, -2, -1, 0, 1, 2, 10\}$
$2x - 10$	has root	$x = 5$	so now	$S = \{-10, -2, -1, 0, 1, 2, 5, 10\}$
$2x + 10$	has root	$x = -5$	so now	$S = \{-10, -5, -2, -1, 0, 1, 2, 5, 10\}$

At this point, no more elements can be added to S. To see this, let

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 = 0$$
$$x(a_n x^{n-1} + a_{n-1}x^{n-2} + \ldots + a_2x + a_1) + a_0 = 0$$
$$x(a_n x^{n-1} + a_{n-1}x^{n-2} + \ldots + a_2x + a_1) = -a_0$$

with each a_i in S. x is a factor of a_0, and a_0 is in S, so x has to be a factor of some element in S. There are no such integers left, so there can be no more additional elements. $\{-10, -5, -2, -1, 0, 1, 2, 5, 10\}$ has 9 elements $\rightarrow \boxed{\textbf{(D)}}$

Problem 22

We let $c, e,$ and m be the probability of reaching a corner before an edge when starting at an "inside corner" (e.g. (1,1)), an "inside edge" (e.g. (1,0)), and the middle respectively. Starting in the middle, there is a 4/8 chance of moving to an inside edge and a 4/8 chance of moving to an inside corner, so

$$m = \frac{1}{2}e + \frac{1}{2}c.$$

Starting at an inside edge, there is a $\frac{2}{8}$ chance of moving to another inside edge, a $\frac{2}{8}$ chance of moving to an inside corner, a $\frac{1}{8}$ chance of moving into the middle, and a $\frac{3}{8}$ chance of reaching an outside edge and stopping. Therefore,

$$e = \frac{1}{4}e + \frac{1}{4}c + \frac{1}{8}m + \frac{3}{8}*0 = \frac{1}{4}e + \frac{1}{4}c + \frac{1}{8}m.$$

Starting at an inside corner, there is a $\frac{2}{8}$ chance of moving to an inside edge, a $\frac{1}{8}$ chance of moving into the middle, a $\frac{4}{8}$ chance of moving to an outside edge and stopping, and finally a $\frac{1}{8}$ chance of reaching that elusive outside corner. This gives

$$c = \frac{1}{4}e + \frac{1}{8}m + \frac{1}{2}0 + \frac{1}{8}*1 = \frac{1}{4}e + \frac{1}{8}m + \frac{1}{8}.$$

Solving this system of equations gives

$$m = \frac{4}{35},$$

$$e = \frac{1}{14},$$

$$c = \frac{11}{70}.$$

Since the particle starts at $(0,0)$, it is m we are looking for, so the final answer is

$$4 + 35 = \boxed{\textbf{(E) } 39}.$$

Problem 23

Solution 1

$f(x)$ must have four roots, three of which are roots of $g(x)$. Using the fact that every polynomial has a unique factorization into its roots, and since the leading coefficient of $f(x)$ and $g(x)$ are the same, we know that

$$f(x) = g(x)(x-r)$$

where $r \in \mathbb{C}$ is the fourth root of $f(x)$. Substituting $g(x)$ and expanding, we find that

$$\begin{aligned} f(x) &= (x^3 + ax^2 + x + 10)(x-r) \\ &= x^4 + (a-r)x^3 + (1-ar)x^2 + (10-r)x - 10r. \end{aligned}$$

Comparing coefficients with $f(x)$, we see that

$$\begin{aligned} a - r &= 1 \\ 1 - ar &= b \\ 10 - r &= 100 \\ -10r &= c. \end{aligned}$$

(Solution 1.1 picks up here.)

Let's solve for $a, b, c,$ and r. Since $10 - r = 100$, $r = -90$, so $c = (-10)(-90) = 900$. Since $a - r = 1$, $a = -89$, and $b = 1 - ar = -8009$. Thus, we know that

$$f(x) = x^4 + x^3 - 8009x^2 + 100x + 900.$$

Taking $f(1)$, we find that

$$\begin{aligned} f(1) &= 1^4 + 1^3 - 8009(1)^2 + 100(1) + 900 \\ &= 1 + 1 - 8009 + 100 + 900 \\ &= \boxed{\textbf{(C)}\ -7007}. \end{aligned}$$

Solution 1.1

A faster ending to Solution 1 is as follows. We shall solve for only a and r. Since $10 - r = 100$, $r = -90$, and since $a - r = 1$, $a = -89$. Then,

$$\begin{aligned} f(1) &= (1+r)(x^3 + ax^2 + x + 10) \\ &= (91)(-77) \\ &= \boxed{\textbf{(C)}\ -7007}. \end{aligned}$$

Solution 2

We notice that the constant term of $f(x) = c$ and the constant term in $g(x) = 10$. Because $f(x)$ can be factored as $g(x)\cdot(x-r)$ (where r is the unshared root of $f(x)$, we see that using the constant term, $-10\cdot r = c$ and therefore $r = -\frac{c}{10}$. Now we once again write $f(x)$ out in factored form:

$$f(x) = g(x)\cdot(x-r) = (x^3+ax^2+x+10)(x+\frac{c}{10})$$

.

We can expand the expression on the right-hand side to get:

$$f(x) = x^4 + (a+\frac{c}{10})x^3 + (1+\frac{ac}{10})x^2 + (10+\frac{c}{10})x + c$$

Now we have
$f(x) = x^4 + (a+\frac{c}{10})x^3 + (1+\frac{ac}{10})x^2 + (10+\frac{c}{10})x + c = x^4 + x^3 + bx^2 + 100x + c.$

Simply looking at the coefficients for each corresponding term (knowing that they must be equal), we have the equations:

$$10 + \frac{c}{10} = 100 \Rightarrow c = 900$$

$$a + \frac{c}{10} = 1, c = 900 \Rightarrow a + 90 = 1 \Rightarrow a = -89$$

and finally,

$$1 + \frac{ac}{10} = b = 1 + \frac{-89\cdot 900}{10} = b = -8009$$

.

We know that $f(1)$ is the sum of its coefficients, hence $1+1+b+100+c$. We substitute the values we obtained for b and c into this expression to get
$f(1) = 1+1+(-8009)+100+900 = \boxed{\textbf{(C)} -7007}$.

Solution 3

Let r_1, r_2, and r_3 be the roots of $g(x)$. Let r_4 be the additional root of $f(x)$. Then from Vieta's formulas on the quadratic term of $g(x)$ and the cubic term of $f(x)$, we obtain the following:

$$r_1 + r_2 + r_3 = -a$$
$$r_1 + r_2 + r_3 + r_4 = -1$$

Thus $r_4 = a - 1$.

Now applying Vieta's formulas on the constant term of $g(x)$, the linear term of $g(x)$, and the linear term of $f(x)$, we obtain:

$$r_1r_2r_3 = -10$$
$$r_1r_2 + r_2r_3 + r_3r_1 = 1$$
$$r_1r_2r_3 + r_2r_3r_4 + r_3r_4r_1 + r_4r_1r_2 = -100$$

Substituting for $r_1r_2r_3$ in the bottom equation and factoring the remainder of the expression, we obtain:

$$-10 + (r_1r_2 + r_2r_3 + r_3r_1)r_4 = -10 + r_4 = -100$$

It follows that $r_4 = -90$. But $r_4 = a - 1$ so $a = -89$

Now we can factor $f(x)$ in terms of $g(x)$ as

$$f(x) = (x - r_4)g(x) = (x + 90)g(x)$$

Then $f(1) = 91g(1)$ and

$$g(1) = 1^3 - 89 \cdot 1^2 + 1 + 10 = -77$$

Hence $f(1) = 91 \cdot (-77) = \boxed{\textbf{(C)} \ -7007}$.

Problem 24

Solution 1

It is easy to see that $\frac{XY}{BD} = 1 - \frac{1}{4} - \frac{11}{36} = \frac{4}{9}$. First we note that $\triangle AXD \sim \triangle EXY$ with a ratio of $\frac{DX}{XY} = \frac{9}{16}$. Then $\triangle ACX \sim \triangle EFX$ with a ratio of $\frac{AX}{XE} = \frac{DX}{XY} = \frac{9}{16}$, so $\frac{XF}{CX} = \frac{16}{9}$. Now we find the length of BD. Because the quadrilateral is cyclic, we can simply use the Law of Cosines.

$$BD^2 = 3^2+8^2-48\cos\angle BAD = 2^2+6^2-24\cos(180-\angle BAD) = 2^2+6^2+24\cos\angle BAD$$

$$\to \cos\angle BAD = \frac{11}{24}$$

$$\to BD = \sqrt{51}$$

By Power of a Point, $CX \cdot XG = DX \cdot XB = \frac{\sqrt{51}}{4}\frac{3\sqrt{51}}{4}$. Thus $XF \cdot XG = \frac{XF}{CX}CX \cdot XG = \frac{51}{3} = \boxed{\textbf{(A)} \ 17}$.

Solution 2

We shall make use of the pairs of similar triangles present in the problem, Ptolemy's Theorem, and Power of a Point. Let Z be the intersection of AC and BD. First, from $ABCD$ being a cyclic quadrilateral, we have that $\triangle BCZ \sim \triangle AZD$, $\triangle BZA \sim \triangle CDZ$. Therefore, $\frac{2}{BZ} = \frac{8}{AZ}$, $\frac{6}{CZ} = \frac{3}{BZ}$, and $\frac{2}{CZ} = \frac{8}{DZ}$, so we have $BZ = \frac{1}{2}CZ$, $AZ = 2CZ$, and $DZ = 4CZ$. By Ptolemy's Theorem,

$$(AB)(CD)+(BC)(DA) = (AC)(BD) = (AZ+ZC)(BZ+ZD)$$

$$\to 3\cdot 6+2\cdot 8 = 34 = (2CZ + ZC)\left(\frac{1}{2}CZ + 4CZ\right) = \frac{27}{2}CZ^2.$$

Thus, $CZ^2 = \frac{68}{27}$. Then, by Power of a Point,
$GX \cdot XC = BX \cdot XD = \frac{3}{4} \cdot \frac{1}{4}BD^2 = \frac{3}{16} \cdot \left(\frac{9}{2}CZ\right)^2 = \frac{9 \cdot 17}{16}$. So, $XG = \frac{9 \cdot 17}{16XC}$. Next, observe that $\triangle ACX \sim \triangle EFX$, so $\frac{XE}{XF} = \frac{AX}{XC}$. Also, $\triangle AXD \sim \triangle EXY$, so $\frac{8}{AX} = \frac{EY}{XE}$. We can compute $EY = \frac{128}{9}$ after noticing that
$XY = BD - BY - DX = BD - \frac{11}{36}BD - \frac{1}{4}BD = \frac{4}{9}BD$ and that
$\frac{8}{DX} = \frac{32}{BD} = \frac{EY}{XY} = \frac{EY}{\frac{4}{9}BD}$. So, $\frac{8}{AX} = \frac{128}{9XE}$. Then, $\frac{XE}{AX} = \frac{XF}{XC} = \frac{16}{9} \rightarrow XF = \frac{16}{9}XC$.
Multiplying our equations for XF and XG yields that
$XF \cdot XG = \frac{9 \cdot 17}{16XC} \cdot \frac{16}{9}XC = \boxed{\textbf{(A) } 17}$.

Problem 25

It is possible to solve this problem using elementary counting methods. This solution proceeds by a cleaner generating function.

We note that $\pm\sqrt{2}i$ both lie on the imaginary axis and each of the $\frac{1}{\sqrt{8}}(\pm 1 \pm i)$ have length $\frac{1}{2}$ and angle of odd multiples of $\pi/4$, i.e. $\pi/4, 3\pi/4, 5\pi, 4, 7\pi/4$. When we draw these 6 complex numbers out on the complex plane, we get a crystal-looking thing. Note that the total number of ways to choose 12 complex numbers is 6^{12}. Now we count the number of good combinations.

We first consider the lengths. When we multiply 12 complex numbers together, their magnitudes multiply. Suppose we have n of the numbers $\pm\sqrt{2}i$; then we must have $\left(\sqrt{2}\right)^n \cdot \left(\frac{1}{2}\right)^{12-n} = 1 \Longrightarrow n = 8$. Having $n = 8$ will take care of the length of the product, now we need to deal with the angle.

We require $\sum \theta \equiv \pi \mod 2\pi$. Letting z be $e^{i\pi/4}$, we see that the angles we have available are $\{z^1, z^2, z^3, z^5, z^6, z^7\}$, where we must choose exactly 8 angles from the set $\{z^2, z^6\}$ and exactly 4 from the set $\{z^1, z^3, z^5, z^7\}$. If we found a good combination where we had a_i of each angle z^i, then the amount this would contribute to our count would be $\binom{12}{4,8} \cdot \binom{8}{a_2, a_6} \cdot \binom{4}{a_1, a_3, a_5, a_7}$. We want to add these all up. We proceed by generating functions.

Consider

$$(t_2x^2 + t_6x^6)^8(t_1x^1 + t_3x^3 + t_5x^5 + t_7x^7)^4.$$

The expansion will be of the form $\sum_i \left(\sum_{\sum a=i} \binom{8}{a_2, a_6}\binom{4}{a_1, a_3, a_5, a_7} t_1{}^{a_1}t_2{}^{a_2}t_3{}^{a_3}t_5{}^{a_5}t_6{}^{a_6}t_7{}^{a_7}x^i\right)$. Note that if we reduced the powers of x mod 8 and fished out the coefficient of x^4 and plugged in $t_i = 1 \ \forall i$ (and then multiplied by $\binom{12}{4,8}$) then we would be done. Since plugging in $t_i = 1$ doesn't affect the x's, we do that right away. The expression then becomes

$$x^{20}(1+x^4)^8(1+x^2+x^4+x^6)^4 = x^{20}(1+x^4)^{12}(1+x^2)^4 = x^4(1+x^4)^{12}(1+x^2)^4,$$

where the last equality is true because we are taking the powers of x mod 8. Let $[x^n]f(x)$ denote the coefficient of x^n in $f(x)$. Note $[x^4]x^4(1+x^4)^{12}(1+x^2)^4 = [x^0](1+x^4)^{12}(1+x^2)^4$. We use the roots of unity filter, which states

$$\text{terms of } f(x) \text{ that have exponent congruent to } k \bmod n = \frac{1}{n}\sum_{m=1}^{n}\frac{f(z^m x)}{z^{mk}},$$

where $z = e^{i\pi/n}$. In our case $k=0$, so we only need to find the average of the $f(z^m x)$'s.

$$z^0 \Longrightarrow (1+x^4)^{12}(1+x^2)^4,$$
$$z^1 \Longrightarrow (1-x^4)^{12}(1+ix^2)^4,$$
$$z^2 \Longrightarrow (1+x^4)^{12}(1-x^2)^4,$$
$$z^3 \Longrightarrow (1-x^4)^{12}(1-ix^2)^4,$$
$$z^4 \Longrightarrow (1+x^4)^{12}(1+x^2)^4,$$
$$z^5 \Longrightarrow (1-x^4)^{12}(1+ix^2)^4,$$
$$z^6 \Longrightarrow (1+x^4)^{12}(1-x^2)^4,$$
$$z^7 \Longrightarrow (1-x^4)^{12}(1-ix^2)^4.$$

We plug in $x=1$ and take the average to find the sum of all coefficients of $x^{\text{multiple of 8}}$. Plugging in $x=1$ makes all of the above zero except for z^0 and z^4. Averaging, we get 2^{14}. Now the answer is simply

$$\frac{\binom{12}{4,8}}{6^{12}} \cdot 2^{14} = \boxed{\frac{2^2\cdot 5\cdot 11}{3^{10}}}.$$

Alternate solution

By changing z_1 to $-z_1$, we can give a bijection between cases where $P=-1$ and cases where $P=1$, so we'll just find the probability that $P=\pm1$ and divide by 2 in the end. Multiplying the hexagon's vertices by i doesn't change P, and switching any z_j with $-z_j$ doesn't change the property $P=\pm1$, so the probability that $P=\pm1$ remains the same if we only select our z_j's at random from

$$\left\{a=\sqrt{2},\quad b=\frac{1}{\sqrt{8}}(1+i),\quad c=\frac{1}{\sqrt{8}}(1-i)\right\}.$$

Since $|a|=\sqrt{2}$ and $|b|=|c|=\frac{1}{2}$, we must choose a exactly 8 times to make $|P|=1$. To ensure P is real, we must either choose b 4 times, c 4 times, or both b and c 2 times. This gives us a total of

$$2\binom{12}{4}+\binom{12}{2}\binom{10}{2} = (12\cdot11\cdot10\cdot9)\left(\frac{1}{12}+\frac{1}{4}\right) = (2^3\cdot3^3\cdot5\cdot11)\frac{1}{3}$$

good sequences $z_1,\ldots,z_{12}$, and hence the final result is

$$\frac{1}{2}\cdot\frac{2^3\cdot3^2\cdot5\cdot11}{3^{12}} = \boxed{(E)}.$$

2017 AMC 12A Answer Key

1. D
2. C
3. B
4. A
5. B
6. B
7. B
8. D
9. E
10. C
11. D
12. B
13. B
14. C
15. D
16. B
17. D
18. D
19. D
20. E
21. D
22. E
23. C
24. A
25. E