2017 AMC 12B Problems and Solutions

Problem 1

Kymbrea's comic book collection currently has 30 comic books in it, and she is adding to her collection at the rate of 2 comic books per month. LaShawn's collection currently has 10 comic books in it, and he is adding to his collection at the rate of 6 comic books per month. After how many months will LaShawn's collection have twice as many comic books as Kymbrea's?

（A）1 （B）4 （C）5 （D）20 （E）25

Problem 2

Real numbers x, y, and z satify the inequalities $0 < x < 1$, $-1 < y < 0$, and $1 < z < 2$. Which of the following numbers is necessarily positive?

$\textbf{(A)}\ y+x^2 \qquad \textbf{(B)}\ y+xz \qquad \textbf{(C)}\ y+y^2 \qquad \textbf{(D)}\ y+2y^2 \qquad \textbf{(E)}\ y+z$

Problem 3

Supposed that x and y are nonzero real numbers such that $\frac{3x+y}{x-3y}=-2$. What is the value of $\frac{x+3y}{3x-y}$?

（A）-3 （B）-1 （C）1 （D）2 （E）3

Problem 4

Samia set off on her bicycle to visit her friend, traveling at an average speed of 17 kilometers per hour. When she had gone half the distance to her friend's house, a tire went flat, and she walked the rest of the way at 5 kilometers per hour. In all it took her 44 minutes to reach her friend's house. In kilometers rounded to the nearest tenth, how far did Samia walk?

（A）2.0 （B）2.2 （C）2.8 （D）3.4 （E）4.4

Problem 5

The data set $[6, 19, 33, 33, 39, 41, 41, 43, 51, 57]$ has median $Q_2 = 40$, first quartile $Q_1 = 33$, and third quartile $Q_3 = 43$. An outlier in a data set is a value that is more than 1.5 times the interquartile range below the first quartile (Q_1) or more than 1.5 times the interquartile range above the third quartile (Q_3), where the interquartile range is defined as $Q_3 - Q_1$. How many outliers does this data set have?

（A）0 （B）1 （C）2 （D）3 （E）4

Problem 6

The circle having (0,0) and (8,6) as the endpoints of a diameter intersects the x-axis at a second point. What is the x-coordinate of this point?

$\textbf{(A)}\ 4\sqrt{2} \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 5\sqrt{2} \qquad \textbf{(D)}\ 8 \qquad \textbf{(E)}\ 6\sqrt{2}$

Problem 7

The functions $\sin(x)$ and $\cos(x)$ are periodic with least period 2π. What is the least period of the function $\cos(\sin(x))$?

$\textbf{(A)}\ \frac{\pi}{2} \qquad \textbf{(B)}\ \pi \qquad \textbf{(C)}\ 2\pi \qquad \textbf{(D)}\ 4\pi \qquad \textbf{(E)}$

Problem 8

The ratio of the short side of a certain rectangle to the long side is equal to the ratio of the long side to the diagonal. What is the square of the ratio of the short side to the long side of this rectangle?

$\textbf{(A)}\ \frac{\sqrt{3}-1}{2} \qquad \textbf{(B)}\ \frac{1}{2} \qquad \textbf{(C)}\ \frac{\sqrt{5}-1}{2} \qquad \textbf{(D)}\ \frac{\sqrt{2}}{2} \qquad \textbf{(E)}\ \frac{\sqrt{6}-1}{2}$

Problem 9

A circle has center (-10,-4) and radius 13. Another circle has center (3,9) and radius $\sqrt{65}$. The line passing through the two points of intersection of the two circles has equation $x+y=c$. What is c?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 3\sqrt{3} \qquad \textbf{(C)}\ 4\sqrt{2} \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ \frac{13}{2}$

Problem 10

At Typico High School, 60% of the students like dancing, and the rest dislike it. Of those who like dancing, 80% say that they like it, and the rest say that they dislike it. Of those who dislike dancing, 90% say that they dislike it, and the rest say that they like it. What fraction of students who say they dislike dancing actually like it?

$\textbf{(A)}\ 10\% \qquad \textbf{(B)}\ 12\% \qquad \textbf{(C)}\ 20\% \qquad \textbf{(D)}\ 25\% \qquad \textbf{(E)}\ 33\frac{1}{3}\%$

Problem 11

Call a positive integer $monotonous$ if it is a one-digit number or its digits, when read from left to right, form either a strictly increasing or a strictly decreasing sequence. For example, 3, 23578, and 987620 are monotonous, but 88, 7434, and 23557 are not. How many monotonous positive integers are there?

（A）1024　（B）1524　（C）1533　（D）1536　（E）2048

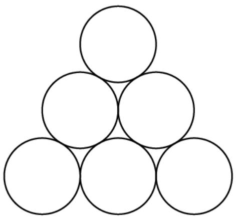
Problem 12

What is the sum of the roots of $z^{12}=64$ that have a positive real part?

$\textbf{(A) } 2 \qquad \textbf{(B) } 4 \qquad \textbf{(C) } \sqrt{2}+2\sqrt{3} \qquad \textbf{(D) } 2\sqrt{2}+\sqrt{6} \qquad \textbf{(E) } (1+\sqrt{3})+(1+\sqrt{3})i$

Problem 13

In the figure below, 3 of the 6 disks are to be painted blue, 2 are to be painted red, and 1 is to be painted green. Two paintings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same. How many different paintings are possible?



（A）6　（B）8　（C）9　（D）12　（E）15

Problem 14

An ice-cream novelty item consists of a cup in the shape of a 4-inch-tall frustum of a right circular cone, with a 2-inch-diameter base at the bottom and a 4-inch-diameter base at the top, packed solid with ice cream, together with a solid cone of ice cream of height 4 inches, whose base, at the bottom, is the top base of the frustum. What is the total volume of the ice cream, in cubic inches?

$\textbf{(A) } 8\pi \qquad \textbf{(B) } \frac{28\pi}{3} \qquad \textbf{(C) } 12\pi \qquad \textbf{(D) } 14\pi \qquad \textbf{(E) } \frac{44\pi}{3}$

Problem 15

Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB'=3AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC'=3BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA'=3CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

（A）9:1　（B）16:1　（C）25:1　（D）36:1　（E）37:1

Problem 16

The number $21!=51,090,942,171,709,440,000$ has over 60,000 positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

$\textbf{(A) } \frac{1}{21} \qquad \textbf{(B) } \frac{1}{19} \qquad \textbf{(C) } \frac{1}{18} \qquad \textbf{(D) } \frac{1}{2} \qquad \textbf{(E) } \frac{11}{21}$

Problem 17

A coin is biased in such a way that on each toss the probability of heads is 2/3 and the probability of tails is 1/3. The outcomes of the tosses are independent. A player has the choice of playing Game A or Game B. In Game A she tosses the coin three times and wins if all three outcomes are the same. In Game B she tosses the coin four times and wins if both the outcomes of the first and second tosses are the same and the outcomes of the third and fourth tosses are the same. How do the chances of winning Game A compare to the chances of winning Game B?

（A） The probability of winning Game A is 4/81 less than the probability of winning Game B.

（B） The probability of winning Game A is 2/81 less than the probability of winning Game B.

（C） The probabilities are the same.

（D） The probability of winning Game A is 2/81 greater than the probability of winning Game B.

（E） The probability of winning Game A is 4/81 greater than the probability of winning Game B.

Problem 18

The diameter AB of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD. Segment AE intersects the circle at a point C between A and E. What is the area of $\triangle ABC$?

$\textbf{(A)}\ \frac{120}{37} \qquad \textbf{(B)}\ \frac{140}{39} \qquad \textbf{(C)}\ \frac{145}{39} \qquad \textbf{(D)}\ \frac{140}{37} \qquad \textbf{(E)}\ \frac{120}{31}$

Problem 19

Let $N = 123456789101112\dots4344$ be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

（A）1 （B）4 （C）9 （D）18 （E）44

Problem 20

Real numbers x and y are chosen independently and uniformly at random from the interval (0,1). What is the probability that $\lfloor\log_2 x\rfloor = \lfloor\log_2 y\rfloor$, where $\lfloor r\rfloor$ denotes the greatest integer less than or equal to the real number r ?

$\textbf{(A)}\ \frac{1}{8} \qquad \textbf{(B)}\ \frac{1}{6} \qquad \textbf{(C)}\ \frac{1}{4} \qquad \textbf{(D)}\ \frac{1}{3} \qquad \textbf{(E)}\ \frac{1}{2}$

Problem 21

Last year Isabella took 7 math tests and received 7 different scores, each an integer between 91 and 100, inclusive. After each test she noticed that the average of her test scores was an integer. Her score on the seventh test was 95. What was her score on the sixth test?

（A）92　（B）94　（C）96　（D）98　（E）100

Problem 22

Abby, Bernardo, Carl, and Debra play a game in which each of them starts with four coins. The game consists of four rounds. In each round, four balls are placed in an urn---one green, one red, and two white. The players each draw a ball at random without replacement. Whoever gets the green ball gives one coin to whoever gets the red ball. What is the probability that, at the end of the fourth round, each of the players has four coins?

$\textbf{(A)}\ \frac{7}{576} \qquad \textbf{(B)}\ \frac{5}{192} \qquad \textbf{(C)}\ \frac{1}{36} \qquad \textbf{(D)}\ \frac{5}{144} \qquad \textbf{(E)}\ \frac{7}{48}$

Problem 23

The graph of $y=f(x)$, where $f(x)$ is a polynomial of degree 3, contains points $A(2,4)$, $B(3,9)$, and $C(4,16)$. Lines AB, AC, and BC intersect the graph again at points D, E, and F, respectively, and the sum of the x-coordinates of D, E, and F is 24. What is $f(0)$?

$\textbf{(A)}\ -2 \qquad \textbf{(B)}\ 0 \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ \frac{24}{5} \qquad \textbf{(E)}\ 8$

Problem 24

Quadrilateral $ABCD$ has right angles at B and C, $\triangle ABC \sim \triangle BCD$, and $AB > BC$. There is a point E in the interior of $ABCD$ such that $\triangle ABC \sim \triangle CEB$ and the area of $\triangle AED$ is 17 times the area of $\triangle CEB$. What is $\frac{AB}{BC}$?

$\textbf{(A)}\ 1+\sqrt{2} \qquad \textbf{(B)}\ 2+\sqrt{2} \qquad \textbf{(C)}\ \sqrt{17} \qquad \textbf{(D)}\ 2+\sqrt{5} \qquad \textbf{(E)}\ 1+2\sqrt{3}$

Problem 25

A set of n people participate in an online video basketball tournament. Each person may be a member of any number of 5-player teams, but no teams may have exactly the same 5 members. The site statistics show a curious fact: The average, over all subsets of size 9 of the set of n participants, of the number of complete teams whose members are among those 9 people is equal to the reciprocal of the average, over all subsets of size 8 of the set of n participants, of the number of complete teams whose members are among those 8 people. How many values n, $9 \leq n \leq 2017$, can be the number of participants?

（A）477　（B）482　（C）487　（D）557　（E）562

2017 AMC 12B Solutions

Problem 1

Kymbrea has 30 comic books initially and every month, she adds two. This can be represented as $30 + 2x$ where x is the number of months elapsed. LaShawn's collection, similarly, is $10 + 6x$. To find when LaShawn will have twice the number of comic books as Kymbrea, we solve for x with the equation $2(2x + 30) = 6x + 10$ and get $x = \boxed{\textbf{(E)}\ 25}$.

Problem 2

Notice that $y + z$ must be positive because $|z| > |y|$. Therefore the answer is $\boxed{\textbf{(E)}\ y + z}$.

The other choices:

$\textbf{(A)}$ As x grows closer to 0, x^2 decreases and thus becomes less than y.

$\textbf{(B)}$ x can be as small as possible $(x > 0)$, so xz grows close to 0 as x approaches 0.

$\textbf{(C)}$ For all $-1 < y < 0$, $y > y^2$, and thus it is always negative.

$\textbf{(D)}$ The same logic as above, but when $-\frac{1}{2} < y < 0$ this time.

Problem 3

Solution 1

Rearranging, we find $3x + y = -2x + 6y$, or $5x = 5y \implies x = y$. Substituting, we can convert the second equation into $\frac{x + 3x}{3x - x} = \frac{4x}{2x} = \boxed{\textbf{(D)}\ 2}$.

Solution 2

Substituting each x and y with 1, we see that the given equation holds true, as $\frac{3(1) + 1}{1 - 3(1)} = -2$. Thus, $\frac{x + 3y}{3x - y} = \boxed{\textbf{(D)}\ 2}$

Problem 4

Let's call the distance that Samia had to travel in total as $2x$, so that we can avoid fractions. We know that the length of the bike ride and how far she walked are equal, so they are both $\frac{2x}{2}$, or x.

She bikes at a rate of 17 kph, so she travels the distance she bikes in $\frac{x}{17}$ hours. She walks at a rate of 5 kph, so she travels the distance she walks in $\frac{x}{5}$ hours.

The total time is $\frac{x}{17}+\frac{x}{5}=\frac{22x}{85}$. This is equal to $\frac{44}{60}=\frac{11}{15}$ of an hour. Solving for x, we have:

$$\frac{22x}{85}=\frac{11}{15}$$
$$\frac{2x}{85}=\frac{1}{15}$$
$$30x=85$$
$$6x=17$$
$$x=\frac{17}{6}$$

Since x is the distance of how far Samia traveled by both walking and biking, and we want to know how far Samia walked to the nearest tenth, we have that Samia walked about $\boxed{(\textbf{C})2.8}$

Problem 5

The interquartile range is defined as $Q3-Q1$, which is $43-33=10$. 1.5 times this value is 15, so all values more than 15 below $Q1$ = $33;-15=18$ is an outlier. The only one that fits this is 6. All values more than 15 above $Q3$ = $43;+15=58$ are also outliers, of which there are none so there is only 1 B.

Problem 6

Because the two points are on a diameter, the center must be halfway between them at the point (4,3). The distance from (0,0) to (4,3) is 5 so the circle has radius 5. Thus, the equation of the circle is $(x-4)^2+(y-3)^2=25$.
To find the x-intercept, y must be 0, so $(x-4)^2+(0-3)^2=25$, so $(x-4)^2=16$, $x=8$.

Problem 7

$\sin(x)$ has values $0,1,0,-1$ at its peaks and x-intercepts. Increase them to $0,\pi/2,0,-\pi/2$. Then we plug them into $\cos(x)$. $\cos(0)=1,\cos(\pi/2)=0,\cos(0)=1$, and $\cos(-\pi/2)=0$. So, $\cos(sin(x))$ is $\frac{2\pi}{2}=\pi\boxed{(\textbf{B})}$

Problem 8

Solution 1: Cross-Multiplication

Let a be the short side of the rectangle, and b be the long side of the rectangle. The diagonal, therefore, is $\sqrt{a^2+b^2}$. We can get the equation $\frac{a}{b}=\frac{b}{\sqrt{a^2+b^2}}$. Cross-multiplying, we get $a\sqrt{a^2+b^2}=b^2$. Squaring both sides of the equation, we get $a^2(a^2+b^2)=b^4$, which simplifies to $a^4+a^2b^2-b^4=0$. Solving for a quadratic in a^2, using the quadratic formula we get $a^2=\frac{-b^2\pm\sqrt{5b^4}}{2}$ which gives us $\frac{a^2}{b^2}=\frac{-1\pm\sqrt{5}}{2}$. We know that the square of the ratio must be positive (the square of any real number is positive), so the solution is $\boxed{\textbf{(C)}\frac{\sqrt{5}-1}{2}}$.

Solution 2: Substitution

Let the short side of the rectangle be a and let the long side of the rectangle be b. Then, the diagonal, according to the Pythagorean Theorem, is $\sqrt{a^2+b^2}$. Therefore, we can write the equation:

$$\frac{a}{b}=\frac{b}{\sqrt{a^2+b^2}}.$$

We are trying to find the square of the ratio of a to b. Let's let our answer, $\frac{a^2}{b^2}$, be k. Then, squaring the above equation,

$$\frac{a^2}{b^2}=k=\frac{b^2}{a^2+b^2}=\frac{b^2}{a^2}-\frac{b^2}{b^2}=\frac{1}{k}-1.$$

Thus, $k=\frac{1}{k}-1$.

Multiplying each side of the equation by k,

$k^2=1-k$.

Adding each side by $k-1$,

$k^2+k-1=0$.

Solving for k using the Quadratic Formula,

$$k=\frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2}=\frac{-1\pm\sqrt{5}}{2}.$$

Since the ratio of lengths and diagonals of a rectangle cannot be negative, and $\sqrt{5}>1$, the $\pm$ symbol can only take on the $+$. Therefore,

$$k=\frac{-1+\sqrt{5}}{2}=\boxed{\textbf{(C)}\frac{\sqrt{5}-1}{2}}.$$

Problem 9

Solution 1

The equations of the two circles are $(x+10)^2+(y+4)^2=169$ and $(x-3)^2+(y-9)^2=65$. Rearrange them to $(x+10)^2+(y+4)^2-169=0$ and $(x-3)^2+(y-9)^2-65=0$, respectively. Their intersection points are where these two equations gain equality. The two points lie on the line with the equation $(x+10)^2+(y+4)^2-169=(x-3)^2+(y-9)^2-65$. We can simplify this like the following.

$(x+10)^2+(y+4)^2-169=(x-3)^2+(y-9)^2-65 \rightarrow (x^2+20x+100)+(y^2+8y+16)-(x^2-6x+9)-(y^2-18y+81)=104 \rightarrow$
$26x+26y+26=104 \rightarrow 26x+26y=78 \rightarrow x+y=3$

. Thus, $c=\boxed{\textbf{(A) } 3}$.

Solution 2: Shortcut with right triangles

Note the specificity of the radii, 13 and $\sqrt{65}$, and that specificity is often deliberately added to simplify the solution to a problem.

One may recognize 13 as the hypotenuse of the 5-12-13 right triangle and $\sqrt{65}$ as the hypotenuse of the right triangle with legs 1 and 8. We can suppose that the legs of these triangles connect the circles' centers to their intersection along the gridlines of the plane.
If we suspect that one of the intersections lies 12 units to the right of and 5 units above the center of the first circle, we find the point (-10+12,-4+5)=(2,1), which is in fact 1 unit to the left of and 8 units below the center of the second circle at (3,9).

Plugging (2,1) into $x+y$ gives us $c=2+1=\boxed{\textbf{(A) } 3}$.
A similar solution uses the other intersection point, (-5,8).

Problem 10

$60\% \cdot 20\% = 12\%$ of the people that claim that they dislike dancing actually like it, and $40\% \cdot 90\% = 36\%$. Therefore, the answer is $\frac{12\%}{12\%+36\%}=\boxed{\textbf{(D) } 25\%}$.

Problem 11

Solution 1

Case 1: monotonous numbers with digits in ascending order

There are $\Sigma_{n=1}^{9}\binom{9}{n}$ ways to choose n digits from the digits 1 to 9. For each of these ways, we can generate exactly one monotonous number by ordering the chosen digits in ascending order. Note that 0 is not included since it will always be a leading digit and that is not allowed. The sum is equivalent to $\Sigma_{n=0}^{9}\binom{9}{n}-\binom{9}{0}=2^9-1=511.$

Case 2: monotonous numbers with digits in descending order

There are $\Sigma_{n=1}^{10}\binom{10}{n}$ ways to choose n digits from the digits 0 to 9. For each of these ways, we can generate exactly one monotonous number by ordering the chosen digits in descending order. Note that 0 is included since we are allowed to end numbers with zeros.

The sum is equivalent to $\Sigma_{n=0}^{10}\binom{10}{n}-\binom{10}{0}=2^{10}-1=1023.$ We discard the number 0 since it is not positive. Thus there are 1022 here.

Since the 1-digit numbers 1 to 9 satisfy both case 1 and case 2, we have overcounted by 9. Thus there are $511+1022-9=\boxed{\textbf{(B)}1524}$ monotonous numbers.

Solution 2

Like Solution 1, divide the problem into an increasing and decreasing case:

- Case 1: Monotonous numbers with digits in ascending order.

Arrange the digits 1 through 9 in increasing order, and exclude 0 because a positive integer cannot begin with 0.

To get a monotonous number, we can either include or exclude each of the remaining 9 digits, and there are $2^9=512$ ways to do this. However, we cannot exclude every digit at once, so we subtract 1 to get $512-1=511$ monotonous numbers for this case.

- Case 2: Monotonous numbers with digits in descending order.

This time, we arrange all 10 digits in decreasing order and repeat the process to find $2^{10}=1024$ ways to include or exclude each digit. We cannot exclude every digit at once, and we cannot include only 0, so we subtract 2 to get 1024-2=1022 monotonous numbers for this case.

At this point, we have counted all of the single-digit monotonous numbers twice, so we must subtract 9 from our total.

Thus our final answer is 511+1022-9=1524 B.

Problem 12

Solution 1

The root of any polynomial of the form $z^n=a$ will have all n of it roots will have magnitude $\sqrt[n]{a}$ and be the vertices of a regular n-gon in the complex plane (This concept is known as the Roots of Unity). For the equation $z^{12}=64$, it is easy to see $\pm\sqrt{2}$ and $\pm i\sqrt{2}$ as roots. Graphing these in the complex plane, we have four vertices of a regular dodecagon. Since the roots must be equally spaced, besides $\sqrt{2}$, there are four more roots with positive real parts lying in the first and fourth quadrants. We also know that the angle between these roots is 30°. We only have to find the real parts of the roots lying in the first quadrant, because the imaginary parts would cancel out with those from the fourth quadrant. We have two $30-60-90$ triangles (the triangles formed by connecting the origin to the roots, and dropping a perpendicular line from each root to the real-axis), both with hypotenuse $\sqrt{2}$. This means that one has base $\frac{\sqrt{2}}{2}$ and the other has base $\frac{\sqrt{6}}{2}$. Adding these and multiplying by two, we get the sum of the four roots as $\sqrt{2}+\sqrt{6}$. However, we have to add in the original solution of $\sqrt{2}$, so the answer is $\boxed{\textbf{(D) } 2\sqrt{2}+\sqrt{6}}$.

Solution 2

$z^{12}=64$ has a factor of $\sqrt{2}$, so we need to remember to multiply our solution below, using the Roots of Unity. We notice that the sum of the complex parts of all these roots is 0, because the points on the complex plane are symmetric. The roots with $re(z)>0$ are $e^{0i}, e^{\frac{\pm\pi}{6}i}$, and $e^{\frac{\pm\pi}{3}i}$ by the Roots of Unity. Their real parts are $\cos(0), \pm\cos(\frac{\pi}{6})$, and $\pm\cos(\frac{\pi}{3})$. Their sum is $1+2(\frac{\sqrt{3}}{2}+\frac{1}{2})=1+\sqrt{3}+1=2+\sqrt{3}$. But, remember to multiply by $\sqrt{2}$. The answer is $\sqrt{2}(2+\sqrt{3})=\boxed{\textbf{(D) } 2\sqrt{2}+\sqrt{6}}$.

Problem 13

Solution 1

First we figure out the number of ways to put the 3 blue disks. Denote the spots to put the disks as 1-6 from left to right, top to bottom. The cases to put the blue disks are $(1,2,3),(1,2,4),(1,2,5),(1,2,6),(2,3,5),(1,4,6)$. For each of those cases we can easily figure out the number of ways for each case, so the total amount is $2+2+3+3+1+1=\boxed{\textbf{(D) } 12}$.

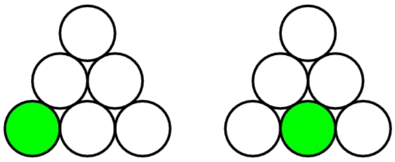
Solution 2

Denote the 6 discs as in the first solution. Ignoring reflections or rotations, there are $\binom{6}{3}*\binom{3}{2}=60$ colorings. Now we need to count the number of fixed points under possible transformations:

1. The identity transformation. Since this doesn't change anything, there are 60 fixed points
2. Reflect about a line of symmetry. There are 3 lines of reflections. Take the line of reflection going through the centers of circles 1 and 5. Then, the colors of circles 2 and 3 must be the same, and the colors of circles 4 and 6 must be the same. This gives us 4fixed points per line of reflection
3. Rotate by 120° counter clockwise or clockwise with respect to the center of the diagram. Take the clockwise case for example. There will be a fixed point in this case if the colors of circles 1,4, and 6 will be the same. Similarly, the colors of circles 2, 3, and 5 will be the same. This is impossible, so this case gives us 0 fixed points per rotation.

By Burnside's lemma, the total number of colorings is $(1*60+3*4+2*0)/(1+3+2)=\boxed{\textbf{(D) } 12}$.

Solution 3

Note that the green disk has two possibilities; in a corner or on the side. WLOG, we can arrange these as 

Take the first case. Now, we must pick two of the five remaining circles to fill in the red. There are $\binom{5}{2} = 10$ of these. However, due to reflection we must divide this by two. But, in two of these cases, the reflection is itself, so we must subtract these out before dividing by 2, and add them back afterwards, giving $\frac{10-2}{2} + 2 = 6$ arrangement in this case.

Now, look at the second case. We again must pick two of the five remaining circles, and like in the first case, two of the reflections give the same arrangement. Thus, there are also 6 arrangements in this case.

In total, we have $6 + 6 = \boxed{\textbf{(D) } 12}$.

Solution 4: Burnside's+Guessing, for if you are running out of time but want some free points

By Burnside's, we quickly find the set G of all group actions that can be applied to one coloring to obtain another. $G = s_1, s_2, s_3, 120^\circ, 240^\circ, e$, where all s_k are lines of reflection symmetry and $120^\circ, 240^\circ$ are the rotations, and e is the identity. Note that all $C(6,3) * C(3,2) = 60$ colorings are fixed by the identity. Let's say there are a total of K other fixed points (that let's say we don't have time to calculate) from the other group actions. Using Burnsides, we have $(60 + K)/6 = 10 + K/6$. At this point, we can clearly get rid of (A), (B), and (C). We have now narrowed down to two answers, and if you're lucky, you can guess $\boxed{\textbf{(D) } 12}$

Problem 14

The top cone has radius 2 and height 4 so it has volume $\frac{1}{3}\pi(2)^2 \times 4$.

The frustum is made up by taking away a small cone of radius 1, height 4 from a large cone of radius 2, height 8, so it has volume $\frac{1}{3}\pi(2)^2 \times 8 - \frac{1}{3}\pi(1)^2 \times 4$.

Adding, we get $\frac{1}{3}\pi(16 + 32 - 4) = \boxed{\textbf{E } \frac{44\pi}{3}}$

Problem 15

Solution 1

Note that by symmetry, $\triangle A'B'C'$ is also equilateral. Therefore, we only need to find one of the sides of $A'B'C'$ to determine the area ratio. WLOG, let $AB = BC = CA = 1$. Therefore, $BB' = 3$ and $BC' = 4$. Also, $\angle B'BC' = 120^\circ$, so by the Law of Cosines, $B'C' = \sqrt{37}$. Therefore, the answer is $(\sqrt{37})^2 : 1^2 = \boxed{\textbf{(E) } 37:1}$

Solution 2

As mentioned in the first solution, $\triangle A'B'C'$ is equilateral. WLOG, let $AB = 2$. Let D be on the line passing through AB such that $A'D$ is perpendicular to AB. Note that $\triangle A'DA$ is a 30-60-90 with right angle at D. Since $AA' = 6$, $AD = 3$ and $A'D = 3\sqrt{3}$. So we know that $DB' = 11$. Note that $\triangle A'DB'$ is a right triangle with right angle at D. So by the Pythagorean theorem, we find $A'B' = \sqrt{(3\sqrt{3})^2 + 11^2} = 2\sqrt{37}$. Therefore, the answer is $(2\sqrt{37})^2 : 2^2 = \boxed{\textbf{(E) } 37:1}$.

Solution 3

Let $AB = BC = CA = x$. We start by noting that we can just write AB' as just $AB + BB' = 4AB$. Similarly $BC' = 4BC$, and $CA' = 4CA$. We can evaluate the area of triangle ABC by simply using Heron's formula, $[ABC] = \sqrt{\frac{3x}{2} \cdot \left(\frac{3x}{2} - x\right)^3} = \frac{x^2\sqrt{3}}{4}$. Next in order to evaluate $A'B'C'$ we need to evaluate the area of the larger triangles $AA'B', BB'C'$, and $CC'A'$. In this solution we shall just compute 1 of these as the others are trivially equivalent. In order to compute the area of $\Delta AA'B'$ we can use the formula $[XYZ] = \frac{1}{2}xy \cdot \sin z$. Since ABC is equilateral and A, B, B' are collinear, we already know $\angle A'AB' = 180 - 60 = 120$ Similarly from above we know AB' and $A'A$ to be $4x$, and $3x$ respectively. Thus the area of $\Delta AA'B'$ is $\frac{1}{2} \cdot 4x \cdot 3x \cdot \sin 120 = 3x^2 \cdot \sqrt{3}$. Likewise we can find $BB'C'$, and $CC'A'$ to also be $3x^2 \cdot \sqrt{3}$.

$$[A'B'C'] = [AA'B'] + [BB'C'] + [CC'A'] + [ABC] = 3 \cdot 3x^2 \cdot \sqrt{3} + \frac{x^2\sqrt{3}}{4} = \sqrt{3} \cdot \left(9x^2 + \frac{x^2}{4}\right)$$

. Therefore the ratio of $[A'B'C']$ to $[ABC]$ is $\frac{\sqrt{3} \cdot \left(9x^2 + \frac{x^2}{4}\right)}{\frac{x^2\sqrt{3}}{4}} = \boxed{\textbf{(E) } 37:1}$

Solution 4 (Elimination)

Looking at the answer choices, we see that all but $\textbf{(E)}$ has a perfect square in the ratio. With some intuition, we can guess that the sidelength of the new triangle formed is not an integer, thus we pick $\boxed{\textbf{(E) } 37:1}$.

Solution 5 (Barycentric Coordinates)

We use barycentric coordinates wrt $\triangle ABC$, to which we can easily obtain that $A' = (4, 0, -3)$, $B' = (-3, 4, 0)$, and $C' = (0, -3, 4)$. Now, since the coordinates are homogenized $(-3 + 4 = 1)$, we can directly apply the area formula to obtain that

$$[A'B'C'] = [ABC] \cdot \begin{vmatrix} 4 & 0 & -3 \\ -3 & 4 & 0 \\ 0 & -3 & 4 \end{vmatrix} = (64-27)[ABC] = 37[ABC],$$

so the answer is $\boxed{\textbf{(E) } 37:1}$

Solution 6 (Area Comparison)

First, comparing bases yields that $[BA'B'] = 3[AA'B] = 9[ABC] \implies [AA'B'] = 12$. By congruent triangles,

$$[AA'B'] = [BB'C'] = [CC'A'] \implies [A'B'C'] = (12+12+12+1)[ABC],$$

which yields that $[A'B'C'] : [ABC] = \boxed{\textbf{(E)}\ 37:1}$

Solution 7 (Quick Proportionality)

Scale down the figure so that the area formulas for the 120° and equilateral triangles become proportional with proportionality constant equivalent to the product of the corresponding sides. By the proportionality, it becomes clear that the answer is $3*4*3+1*1 = 37:1, \boxed{\text{E}}$. ~ Solution by mathchampion1

Problem 16

Solution1

We note that the only thing that affects the parity of the factor are the powers of 2. There are 10+5+2+1=18 factors of 2 in the number. Thus, there are 18 cases in which a factor of $21!$ would be even (have a factor of 2 in its prime factorization), and 1 case in which a factor of $21!$ would be odd. Therefore, the answer is $\boxed{\textbf{(B)} \frac{1}{19}}$

Solution 2: Constructive counting

Solution 2: Constructive counting

Consider how to construct any divisor D of $21!$. First by Legendre's theorem for the divisors of a factorial (see here: http://www.cut-the-knot.org/blue/LegendresTheorem.shtml and here: Legendre's Formula), we have that there are a total of 18 factors of 2 in the number. D can take up either 0, 1, 2, 3, . . . , or all 18 factors of 2, for a total of 19 possible cases. In order for D to be odd, however, it must have 0 factors of 2, meaning that there is a probability of 1 case/19 cases= $\boxed{1/19, \text{B}}$

Problem 17

The probability of winning Game A is the sum of the probabilities of getting three tails and getting three heads which is $\left(\frac{2}{3}\right)^3 + \left(\frac{1}{3}\right)^3 = \frac{8}{27} + \frac{1}{27} = \frac{9}{27} = \frac{27}{81}$. The probability of winning Game B is the sum of the probabilities of getting two heads and getting two tails squared. This gives us $\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right)^2 = \left(\frac{5}{9}\right)^2 = \frac{25}{81}$. The probability of winning Game A is $\frac{27}{81}$ and the probability of winning Game B is $\frac{25}{81}$, so the answer is $\boxed{\textbf{(D)}}$

Problem 18

Solution1

Notice that ADE and ABC are right triangles. Then $AE=\sqrt{7^2+5^2}=\sqrt{74}$. $\sin DAE=\frac{5}{\sqrt{74}}=\sin BAE=\sin BAC=\frac{BC}{4}$, so $BC=\frac{20}{\sqrt{74}}$. We also find that $AC=\frac{28}{\sqrt{74}}$, and thus the area of ABC is $\frac{\frac{20}{\sqrt{74}}\cdot\frac{28}{\sqrt{74}}}{2}=\frac{\frac{560}{74}}{2}=\boxed{\textbf{(D)}\ \frac{140}{37}}$.

Solution2

We note that $\triangle ACB\ \triangle ADE$ by AA similarity. Also, since the area of $\triangle ADE=\frac{7\cdot 5}{2}=\frac{35}{2}$ and $AE=\sqrt{74}$, $\frac{[ABC]}{[ADE]}=\frac{[ABC]}{\frac{35}{2}}=\left(\frac{4}{\sqrt{74}}\right)^2$, so the area of $\triangle ABC=\boxed{\textbf{(D)}\ \frac{140}{37}}$.

Solution3

As stated before, note that $\triangle ACB\ \triangle ADE$. By similarity, we note that $\frac{\overline{AC}}{\overline{BC}}$ is equivalent to $\frac{7}{5}$. We set $\overline{AC}$ to $7x$ and $\overline{BC}$ to $5x$. By the Pythagorean Theorem, $(7x)^2+(5x)^2$ = 4^2. Combining, $49x^2+25x^2=16$. We can add and divide to get $x^2=\frac{8}{37}$. We square root and rearrange to get $x=\frac{2\sqrt{74}}{37}$. We know that the legs of the triangle are $7x$ and $5x$. Mulitplying x by 7 and 5 eventually gives us $\frac{14\sqrt{74}}{37}$x$\frac{10\sqrt{74}}{37}$. We divide this by 2, since $\frac{1}{2}bh$ is the formula for a triangle. This gives us $\boxed{\textbf{(D)}\ \frac{140}{37}}$.

Solution4

Let's call the center of the circle that segment AB is the diameter of, O. Note that $\triangle ODE$ is an isosceles right triangle. Solving for side OE, using the Pythagorean theorem, we find it to be $5\sqrt{2}$. Calling the point where segment OE intersects circle O, the point I, segment IE would be $5\sqrt{2}-2$. Also, noting that $\triangle ADE$ is a right triangle, we solve for side AE, using the Pythagorean Theorem, and get $\sqrt{74}$. Using Power of Point on point E, we can solve for CE. We can subtract CE from AE to find AC and then solve for CB using Pythagorean theorem once more.

$(AE)(CE)$ = (Diameter of circle O + $IE)(IE)->\sqrt{74}(CE)$ = $(5\sqrt{2}+2)(5\sqrt{2}-2)->$ CE = $\frac{23\sqrt{74}}{37}$

$AC=AE-CE->AC$ = $\sqrt{74}$ - $\frac{23\sqrt{74}}{37}$ $->AC$ = $\frac{14\sqrt{74}}{37}$

Now to solve for CB:

AB^2 - AC^2 = $CB^2->4^2$ + $\frac{14\sqrt{74}}{37}^2$ = $CB^2->CB$ = $\frac{10\sqrt{74}}{37}$

Note that $\triangle ABC$ is a right triangle because the hypotenuse is the diameter of the circle. Solving for area using the bases AC and BC, we get the area of triangle ABC to be $\boxed{\textbf{(D)}\ \frac{140}{37}}$.

Problem 19

Solution1

We only need to find the remainders of N when divided by 5 and 9 to determine the answer. By inspection, $N \equiv 4 \pmod{5}$. The remainder when N is divided by 9 is $1+2+3+4+\cdots+1+0+1+1+1+2+\cdots+4+3+4+4$, but since $10 \equiv 1 \pmod{9}$, we can also write this as $1+2+3+\cdots+10+11+12+\cdots 43+44 = \frac{44 \cdot 45}{2} = 22 \cdot 45$, which has a remainder of 0 mod 9. Therefore, by inspection, the answer is $\boxed{\textbf{(C)}\ 9}$.

Note: the sum of the digits of N is 270.

Solution2

Noting the solution above, we try to find the sum of the digits to figure out its remainder when divided by 9. From 1 thru 9, the sum is 45. 10 thru 19, the sum is 55, 20 thru 29 is 65, and 30 thru 39 is 75. Thus the sum of the digits is $45+55+65+75+4+5+6+7+8 = 240+30 = 270$, and thus N is divisible by 9. Now, refer to the above or below solutions. $N \equiv 4 \pmod{5}$ and $N \equiv 0 \pmod{9}$. From this information, we can conclude that $N \equiv 54 \pmod{5}$ and $N \equiv 54 \pmod{9}$. Therefore, $N \equiv 54 \pmod{45}$ and $N \equiv 9 \pmod{45}$ so the remainder is $\boxed{\textbf{(C)}\ 9}$

Problem 20

First let us take the case that $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor = -1$. In this case, both x and y lie in the interval $[1/2, 1)$. The probability of this is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Similarly, in the case that $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor = -2$, x and y lie in the interval $[1/4, 1/2)$, and the probability is $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$. It is easy to see that the probabilities for $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor = n$ for $-\infty < n < 0$ are the infinite geometric series that starts at $\frac{1}{4}$ and with common ratio $\frac{1}{4}$. Using the formula for the sum of an infinite geometric series, we get that the probability is $\frac{\frac{1}{4}}{1-\frac{1}{4}} = \boxed{\textbf{(D)} \frac{1}{3}}$.

Problem 21

Solution 1

Let the sum of the scores of Isabella's first 6 tests be S. Since the mean of her first 7 scores is an integer, then $S + 95 \equiv 0 \pmod{7}$, or $S \equiv 3 \pmod{7}$. Also, $S \equiv 0 \pmod{6}$, so by CRT, $S \equiv 24 \pmod{42}$. We also know that $91 \cdot 6 \leq S \leq 100 \cdot 6$, so by inspection, $S = 570$. However, we also have that the mean of the first 5 integers must be an integer, so the sum of the first 5 test scores must be an multiple of 5, which implies that the 6th test score is $\boxed{\textbf{(E)}\ 100}$.

Solution 2

First, we find the largest sum of scores which is $100 + 99 + 98 + 97 + 96 + 95 + 94$ which equals $7(97)$. Then we find the smallest sum of scores which is $91 + 92 + 93 + 94 + 95 + 96 + 97$ which is $7(94)$. So the possible sums for the 7 test scores so that they provide an integer average are $7(97), 7(96), 7(95)$ and $7(94)$ which are $679, 672, 665,$ and 658 respectively. Now in order to get the sum of the first 6 tests, we negate 95 from each sum producing $584, 577, 570,$ and 563. Notice only 570 is divisible by 6 so, therefore, the sum of the first 6 tests is 570. We need to find her score on the $6th$ test so what number minus 570 will give us a number divisible by 5. Since 95 is the $7th$ test score and all test scores are distinct that only leaves $\boxed{\textbf{(E)}\ 100}$.

Solution 3

By inspection, the sequences $91, 93, 92, 96, 98, 100, 95$ and $93, 91, 92, 96, 98, 100, 95$ work, so the answer is $\boxed{\textbf{(E)}\ 100}$. Note: A method of finding this "cheap" solution is to create a "mod chart", basically list out the residues of 91-100 modulo 1-7 and then finding the two sequences should be made substantially easier.

Problem 22

Solution 1

It amounts to filling in a 4×4 matrix. Columns $C_1 - C_4$ are the random draws each round; rows $R_A - R_D$ are the coin changes of each player. Also, let $\%R_A$ be the number of nonzero elements in R_A.

WLOG, let $C_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$. Parity demands that $\%R_A$ and $\%R_B$ must equal 2 or 4.

Case 1: $\%R_A = 4$ and $\%R_B = 4$. There are $\binom{3}{2} = 3$ ways to place 2 -1's in R_A, so there are 3 ways.

Case 2: $\%R_A = 2$ and $\%R_B = 4$. There are 3 ways to place the -1 in R_A, 2 ways to place the remaining -1 in R_B (just don't put it under the -1 on top of it!), and 2 ways for one of the other two players to draw the green ball. (We know it's green because Bernardo drew the red one.) We can just double to cover the case of $\%R_A = 4$, $\%R_B = 2$ for a total of 24 ways.

Case 3: $\%R_A = \%R_B = 2$. There are three ways to place the -1 in R_A. Now, there are two cases as to what happens next.

Sub-case 3.1: The 1 in R_B goes directly under the -1 in R_A. There's obviously 1 way for that to happen. Then, there are 2 ways to permute the two pairs of $1, -1$ in R_C and R_D. (Either the 1 comes first in R_C or the 1 comes first in R_D.)

Sub-case 3.2: The 1 in R_B doesn't go directly under the -1 in R_A. There are 2 ways to place the 1, and 2 ways to do the same permutation as in Sub-case 3.1. Hence, there are $3(2 + 2 \cdot 2) = 18$ ways for this case.

There's a grand total of 45 ways for this to happen, along with 12^3 total cases. The probability we're asking for is thus $\frac{45}{(12^3)} = \frac{5}{192}$.

Solution 2

We will proceed by taking cases based on how many people are taking part in this "transaction." We can have 2, 3, or 4 people all giving/receiving coins during the 4 turns. Basically, (like the previous solution), we are thinking this as filling out a 4x2 matrix of letters, where a letter on the left column represents this person gave, and a letter on the right column means this person achieved. We need to make sure that for each person that gave a certain amount, they received in total from other people that same amount, or in other words there are an equal number of A's, B's, C's, and D's on both columns of the matrix.

Case 1: 2 people. In this case, we can 4C2 ways to choose the two people, and 6 ways to get order them to get a count of $6 * 6 = 36$ ways.

Case 2: 3 people. In this case, we have 4*(3C2)*4! = 288 ways to order 3 people.

Case 3: 4 people. In this case, we have 3*3*4! = 216 ways to order 4 people.

So we have a total of 36+288+126=540 ways to order the four pairs of people. Now we divide this by the total number of ways - (4*3)^4 (4 times, 4 ways to choose giver, 3 to choose receiver). So the answer is 5/192.

~ccx09 (NOTE: Due to the poor quality of this solution, please PM me and I will explain the numbers, I have some diagrams but I can't show it here)

Problem 23

First, we can define $f(x) = a(x-2)(x-3)(x-4) + x^2$, which contains points A, B, and C. Now we find that lines AB, AC, and BC are defined by the equations $y = 5x - 6$, $y = 6x - 8$, and $y = 7x - 12$ respectively. Since we want to find the x-coordinates of the intersections of these lines and $f(x)$, we set each of them to $f(x)$, and synthetically divide by the solutions we already know exist (eg. if we were looking at line AB, we would synthetically divide by the solutions $x = 2$ and $x = 3$, because we already know AB intersects the graph at A and B, which have x-coordinates of 2 and 3). After completing this process on all three lines, we get that the x-coordinates of D, E, and F are $\frac{4a-1}{a}$, $\frac{3a-1}{a}$, and $\frac{2a-1}{a}$ respectively. Adding these together, we get $\frac{9a-3}{a} = 24$ which gives us $a = -\frac{1}{5}$. Substituting this back into the original equation, we get $f(x) = -\frac{1}{5}(x-2)(x-3)(x-4) + x^2$, and $f(0) = -\frac{1}{5}(-2)(-3)(-4) + 0 = \boxed{\textbf{(D)}\,\frac{24}{5}}$

Problem 24

Solution 1

Let $CD = 1$, $BC = x$, and $AB = x^2$. Note that $AB/BC = x$. By the Pythagorean Theorem, $BD = \sqrt{x^2+1}$. Since $\triangle BCD \sim \triangle ABC \sim \triangle CEB$, the ratios of side lengths must be equal. Since $BC = x$, $CE = \frac{x^2}{\sqrt{x^2+1}}$ and $BE = \frac{x}{\sqrt{x^2+1}}$. Let F be a point on $\overline{BC}$ such that $\overline{EF}$ is an altitude of triangle CEB. Note that $\triangle CEB \sim \triangle CFE \sim \triangle EFB$. Therefore, $BF = \frac{x}{x^2+1}$ and $CF = \frac{x^3}{x^2+1}$. Since $\overline{CF}$ and $\overline{BF}$ form altitudes of triangles CED and BEA, respectively, the areas of these triangles can be calculated. Additionally, the area of triangle BEC can be calculated, as it is a right triangle. Solving for each of these yields:

$$[BEC] = [CED] = [BEA] = (x^3)/(2(x^2+1))$$

$$[ABCD] = [AED] + [DEC] + [CEB] + [BEA]$$

$$(AB+CD)(BC)/2 = 17*[CEB]+[CEB]+[CEB]+[CEB]$$

$$(x^3+x)/2 = (20x^3)/(2(x^2+1))$$

$$(x)(x^2+1) = 20x^3/(x^2+1)$$

$$(x^2+1)^2 = 20x^2$$

$$x^4 - 18x^2 + 1 = 0 \implies x^2 = 9 + 4\sqrt{5} = 4 + 2(2\sqrt{5}) + 5$$

Therefore, the answer is $\boxed{\textbf{(D) } 2+\sqrt{5}}$

Solution 2

Draw line FG through E, with F on BC and G on AD, $FG \parallel AB$. WLOG let $CD = 1$, $CB = x$, $AB = x^2$. By weighted average $FG = \frac{1+x^4}{1+x^2}$.

Meanwhile, $FE : EG = [\triangle CBE] : [\triangle ADE] = 1 : 17$.

$FE = \frac{x^2}{1+x^2}$. We obtain $\frac{1+x^4}{1+x^2} = \frac{18x^2}{1+x^2}$, namely $x^4 - 18x^2 + 1 = 0$.

The rest is the same as Solution 1.

Solution 3

Let $A = (-1, 4a), B = (-1, 0), C = (1, 0), D = \left(1, \frac{1}{a}\right)$. Then from the similar triangles condition, we compute $CE = \frac{4a}{\sqrt{4a^2+1}}$ and $BE = \frac{2}{\sqrt{4a^2+1}}$. Hence, the y-coordinate of E is just $\frac{BE \cdot CE}{BC} = \frac{4a}{4a^2+1}$. Since E lies on the unit circle, we can compute the x coordinate as $\frac{1-4a^2}{4a^2+1}$. By Shoelace, we want

$$\frac{1}{2}\det\begin{bmatrix} -1 & 4a & 1 \\ \frac{1-4a^2}{4a^2+1} & \frac{4a}{4a^2+1} & 1 \\ 1 & \frac{1}{a} & 1 \end{bmatrix} = 17 \cdot \frac{1}{2} \cdot 2 \cdot \frac{4a}{4a^2+1}$$

Factoring out denominators and expanding by minors, this is equivalent to

$$\frac{32a^4 - 8a^2 + 2}{2a(4a^2+1)} = \frac{68a}{4a^2+1} \implies 16a^4 - 72a^2 + 1 = 0$$

This factors as $(4a^2 - 8a - 1)(4a^2 + 8a - 1) = 0$, so $a = 1 + \frac{\sqrt{5}}{2}$ and so the answer is $\textbf{(D)}$

Problem 25

Let there be T teams. For each team, there are $\binom{n-5}{4}$ different subsets of 9 players including that full team, so the total number of team-(group of 9) pairs is

$$T\binom{n-5}{4}.$$

Thus, the expected value of the number of full teams in a random set of 9 players is

$$\frac{T\binom{n-5}{4}}{\binom{n}{9}}.$$

Similarly, the expected value of the number of full teams in a random set of 8 players is

$$\frac{T\binom{n-5}{3}}{\binom{n}{8}}.$$

The condition is thus equivalent to the existence of a positive integer T such that

$$\frac{T\binom{n-5}{4}}{\binom{n}{9}}\frac{T\binom{n-5}{3}}{\binom{n}{8}} = 1.$$

$$T^2\frac{(n-5)!(n-5)!8!9!(n-8)!(n-9)!}{n!n!(n-8)!(n-9)!3!4!} = 1$$

$$T^2 = \big((n)(n-1)(n-2)(n-3)(n-4)\big)^2\frac{3!4!}{8!9!}$$

$$T^2 = \big((n)(n-1)(n-2)(n-3)(n-4)\big)^2\frac{144}{7!7!8\cdot 8\cdot 9}$$

$$T^2 = \big((n)(n-1)(n-2)(n-3)(n-4)\big)^2\frac{1}{4\cdot 7!7!}$$

Note that this is always less than $\binom{n}{5}$, so as long as T is integral, n is a possibility. Thus, we have that this is equivalent to

$$2^5\cdot 3^2\cdot 5\cdot 7\big|(n)(n-1)(n-2)(n-3)(n-4).$$

It is obvious that 5 divides the RHS, and that 7 does iff $n\equiv 0,1,2,3,4 \mod 7$. Also, 3^2 divides it iff $n\not\equiv 5,8 \mod 9$. One can also bash out that 2^5 divides it in 16 out of the 32 possible residues $\mod 32$.

Using all numbers from 2 to 2017, inclusive, it is clear that each possible residue $\mod 7,9,32$ is reached an equal number of times, so the total number of working n in that range is $5\cdot 7\cdot 16 = 560$. However, we must subtract the number of "working" $2\leq n\leq 8$, which is 3. Thus, the answer is $\boxed{\textbf{(D) } 557}$.

2017 AMC 12B Answer Key

1. E
2. E
3. D
4. C
5. B
6. D
7. B
8. C
9. A
10. D
11. B
12. D
13. D
14. E
15. E
16. B
17. D
18. D
19. C
20. D
21. E
22. B
23. D
24. D
25. D